

Topics in the Heavy Quark Expansion

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Abstract

Achievements in the heavy quark theory over the last decade are reviewed, with the main emphasis put on dynamical methods which quantify nonperturbative effects via application of the Operator Product Expansion. These include the total weak decay rates of heavy flavor hadrons and nonperturbative corrections to heavy quark sum rules. Two new exact superconvergent sum rules are derived; they differ from the known ones in that they are finite and normalization point independent in perturbation theory. A new hadronic parameter $\bar{\Sigma}$ is introduced which is a spin-nonsinglet analogue of $\bar{\Lambda} = M_B - m_b$; it is expected to be about 0.25 GeV. The first sum rule implies the bound $\rho^2 > 3/4$ for the slope of the Isgur-Wise function. The heavy quark potential is discussed and its connection to the infrared contributions in the heavy quark mass. Among applications extraction of $|V_{cb}|$ from the total semileptonic and from the $B \rightarrow D^*$ zero recoil rates is addressed, as well as extracting $|V_{ub}|$ from $\Gamma_{sl}(b \rightarrow u)$. Practical aspects of local quark-hadron duality are briefly discussed.

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1 Introduction

Quark-gluon dynamics are governed by the QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a + \sum_{\psi} \bar{\psi}(i\not{D} - m_{\psi})\psi \quad (1)$$

where $G_{\mu\nu}^a$ is the gluon field strength tensor, and ψ are quark fields. Masses of u and d quarks are only a few MeV, much smaller than Λ_{QCD} , and in most applications of hadron physics u and d can be considered as massless. The strange quark mass is about 150 MeV, only a little smaller than Λ_{QCD} . Nevertheless, there is ample evidence that treating s quark as light is justified, and corrections to the (light) $SU(3)$ symmetry are reasonably small. Thus, m_s lies essentially below the actual typical hadronic QCD scale $\mu_{\text{hadr}} \sim 500$ to 700 MeV.

Those quarks Q for which $m_Q \gg \mu_{\text{hadr}} \sim (2-3)\Lambda_{\text{QCD}}$ are heavy quarks. The sixth t quark is the heaviest, $m_t \approx 170$ GeV. However, it is too heavy. Its width due to the ‘semiweak’ decay $t \rightarrow b + W^+$ is $\Gamma_t \approx 1$ GeV. It decays too fast for the t -hadrons, the bound states with light quarks to be formed.

The best candidates for application of the Heavy Quark Expansion (HQE) are beauty hadrons. b quark is heavy enough to confidently use the expansion in $1/m_b$, yet the leading corrections to the heavy quark limit are not negligible.

The charmed quark c can be called heavy only with some reservations. While in some cases this yields a reasonable approximation, m_c often appears manifestly too low for a quantitative treatment of charmed physics in the $1/m_Q$ expansion. There is no universal answer here, and considerable care must be exercised in every particular case.

Usually considered heavy flavor hadrons are composed from one heavy quark Q , and a light cloud: a light antiquark \bar{q} , or diquark qq , together with the gluon medium and light quark-antiquark pairs. The role of the gluon medium is to keep all ‘valence’ constituents together, in a colorless bound state which will be generically denoted by H_Q . Therefore, we have the following simplified picture of a heavy flavor hadron. The heavy quark has a small size $\sim 1/m_Q$, and is surrounded by a static Coulomb-like color field A_0 at small distances. Non-Abelian selfinteraction slightly modifies the potential, but the nonlinearity is driven by the coupling $\alpha_s(r^{-1})/\pi$ and is not significant. At larger distances the selfinteraction strengthens, at $R \gtrsim \Lambda_{\text{QCD}}^{-1}$ it is completely nonperturbative: the soft modes of the light fields are strongly coupled and strongly fluctuate.

In weak decays the standard situation is that the external (to QCD) forces like W bosons interact with the heavy quark, say, instantaneously replace the b quark by c quark generally changing its velocity. Such an event excites first the typical modes of the heavy quark, hard gluons with $\vec{k}_{\text{typ}} \sim m_Q$. Since

$$\alpha_s(k_{\text{typ}}) \sim \alpha_s(m_Q) \ll 1$$

perturbation theory is adequate there.

An actual decay process, nevertheless, eventually runs into the strong-interaction nonperturbative domain of $\vec{k} \sim \omega \sim \Lambda_{\text{QCD}}$, where ω denotes the characteristic frequencies. The final hadronization dynamics shaping the hadrons observed in experiment, is a result of soft nonperturbative physics which is responsible for confinement. Since the complicated final state dynamics involve $\omega \ll m_Q$, to deal with nonperturbative effects one can use the *nonrelativistic expansion* for the heavy quark.

The main subject of the HQE is nonperturbative physics. First, the perturbative corrections are conceptually simple, even if the actual computations are often cumbersome and, beyond the first-order effects, require sophisticated state-of-the-art technique.

Second, the perturbative corrections are calculated in the full QCD rather than in the effective low-energy theory, since they come just from the gluon momenta $k \sim m_Q$ where the nonrelativistic approximation is not applicable. Still, the interplay of the perturbative and nonperturbative effects is quite nontrivial and often involves theoretical subtleties.

The treatment of the nonperturbative effects is a nontrivial problem, and different methods of QCD are used here. The basic tool for all of them in heavy quarks is the Wilson operator product expansion (OPE) [1].

1.1 Nonrelativistic expansion

The main simplification of a nonrelativistic treatment is that the number of heavy quarks n_Q and antiquarks \bar{n}_Q are separately conserved. Propagation with usual relativistic Green function for the heavy quark contains also the process of the $Q\bar{Q}$ pair creation if the time ordering of the vertices along the line is reversed somewhere. In the nonrelativistic kinematics, however, such configurations yield a power-suppressed contribution, since the virtuality of the intermediate state (the energy denominator, in the language of noncovariant perturbation theory of quantum mechanics) is of the order of $2m_Q$. In order to observe such processes as real ones, it would be necessary to supply energy at least as large as $\omega \approx 2m_Q$.

Heavy quarks can appear also in closed loops, for example, in the processes of the virtual gluon conversion. For heavy quarks such effects are also suppressed, $\sim \vec{k}^2/m_Q^2$ if the gluon momentum k is much smaller than m_Q .

As a result, the field-theoretic, or second-quantized description of the heavy quark becomes redundant, and it is sufficient to resort to its usual quantum-mechanical (QM) treatment. For example, the wavefunction of a heavy flavor hadron takes the form

$$\Psi_\alpha [\vec{x}_Q, \{x_{\text{light}}\}] \quad , \quad (2)$$

where \vec{x}_Q is the heavy quark coordinate, x_{light} generically represents an infinite number of light degrees of freedom; index α describes the heavy quark spin.

A relativistic $S = \frac{1}{2}$ particle has four components, i.e. $\alpha = 1..4$. The non-relativistic spinor $\Psi(x)$ has only two of them, $\alpha = 1, 2$ describing two spin states. The remaining $\alpha = \{3, 4\}$ components describe antiparticles which decouple in the

nonrelativistic theory:

$$Q(x) = \begin{pmatrix} \Psi(x) \\ \chi(x) \end{pmatrix} \quad \begin{aligned} \Psi(x) &\sim \mathcal{O}(1) \\ \chi(x) &\sim \frac{\vec{p}}{m_Q} \rightarrow 0 \end{aligned} \quad (3)$$

The nonrelativistic Hamiltonian of the spin- $\frac{1}{2}$ particle is the well known Pauli Hamiltonian:¹

$$\mathcal{H}_{\text{Pauli}} = -A_0 + \frac{(i\vec{\partial} - \vec{A})^2}{2m} + \frac{\vec{\sigma}\vec{B}}{2m}. \quad (4)$$

The last operator is written with the coefficient appearing for an ‘elementary’ point-like particle. Presence of the interaction generally renormalizes its strength, the chromomagnetic moment of the heavy quark. In the heavy quark limit $m_Q \rightarrow \infty$ only the first term survives while the last two terms describing space propagation and interaction of spin with the chromomagnetic field \vec{B} , disappear. An infinitely heavy quark is static and interacts only with the color Coulomb potential.² In turn, the heavy quark is the source of a static color field independent of the heavy quark spin. The actual dynamics of the heavy quark spin reveal itself in full only when the quark interacts with quanta having large momentum $|\vec{q}| \gtrsim m_Q$.

It is easy to illustrate the b quark spin-independence of the strong forces in the following way. One can imagine the QCD world where, instead of the actual b quark there exists a scalar spinless \tilde{b} quark with the same mass. The usual spin-0 B meson, $b\bar{q}$ would become a spin- $\frac{1}{2}$ particle $\tilde{B} \sim \tilde{b}\bar{q}$. The Λ_b baryon bud having spin- $\frac{1}{2}$ would, in turn become a scalar spinless \tilde{bud} state; Σ_b would have spin 1. Nevertheless, at $m_b \rightarrow \infty$ the properties of B and \tilde{B} or Λ_b and $\tilde{\Lambda}_b$ would be identical. The two degenerate spin states of \tilde{B} are counterparts of B and B^* of the actual QCD.

An immediate question comes to one’s mind at this point: what about the relation between spin and statistics? In such a gedanken operation we replace spin- $\frac{1}{2}$ hadrons by $S = 0$ ones, and *vice versa*, *i.e.* interchange fermions and bosons. It is clear, however, that for the states or processes with a single heavy quark the statistics symmetry properties do not play a role.

Such an independence of the strong dynamics of the heavy quark spin is called the heavy quark *Spin Symmetry*.

The rest mass m_Q of the heavy quark also enters in a trivial way: the Hamiltonian simply contains $(n_Q + \bar{n}_Q)m_Q$. Since both are fixed, it is an overall additive constant. For a moving quark this constant is $E = \sqrt{m_Q^2 + \vec{p}^2} = m_Q\sqrt{1 + \vec{v}^2}$. Therefore, the actual dynamics is not affected by the concrete value of the mass m_Q . This is the *Heavy Flavor Symmetry* which states, for example, the equal properties of charmed and beauty hadrons to the extent they both can be considered heavy enough.

The heavy flavor symmetry leads also to certain scaling behavior of the transition amplitudes with heavy flavor hadrons: the amplitudes depend on their velocities

¹We use the convention where the coupling g_s is absorbed in the gauge fields; this is convenient for nonperturbative analysis.

²Speaking of the Coulomb interaction in QCD we mean the interaction with the timelike component A_0 of the color gauge potential. It differs from the simple $1/R$ electrodynamic potential.

rather than on the absolute values of momenta:

$$A(P_Q^{in}, p_i^{in}; P_{Q'}^{out}, p_l^{out}) = \mathcal{A}\left(\frac{P_Q^{in}}{m_Q}, p_i^{in}; \frac{P_{Q'}^{out}}{m_{Q'}}, p_l^{out}\right). \quad (5)$$

Here P denote the momenta of the heavy flavor hadrons and p refer to other participating particles. Such a scaling behavior is valid only with respect to the soft part of the interaction, and no hard gluons with $\vec{k} \sim m_Q$ are involved.

The one-particle (or QM) description of the heavy quark degrees of freedom is the key simplification of the nonrelativistic expansion. The light cloud, however, still requires a full-fledged field-theoretic treatment. Even considering the static limit $m_Q \rightarrow \infty$ where only interaction with the Coulomb field A_0 remains, the latter strongly fluctuates, in contrast to simple potential QM models where the corresponding potential $V(x)$ is a c -number function of coordinates.

For this reason even the quantum mechanics of heavy quarks is highly nontrivial. Exploiting the symmetry properties of the heavy quark interactions does not require understanding of these complicated strong interaction dynamics. This was the main field of applications at an early stage of theoretical development of heavy quark physics. The recent progress is mainly related to a better treatment of basic properties of this complicated strongly interacting system via application of dynamic QCD methods based on the short distance expansion.

1.2 Operator Product Expansion

The basic theoretical tool of the heavy quark theory is the Wilson operator product expansion [1]. The idea of the OPE was formulated by K. Wilson in the late 60's, originally in the context of the statistical problems which are closely related to the field theories in Euclidean space. This idea, in general, is a separation of effects originating at large and small distances. The application to real physical processes in Minkowski space is often less transparent and technically more complicated, however it is always based on the same underlying concept.

The original QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a + \sum_q \bar{q}(i\not{D}-m_q)q + \sum_Q \bar{Q}(i\not{D}-m_Q)Q = \mathcal{L}_{\text{light}} + \sum_Q \bar{Q}(i\not{D}-m_Q)Q \quad (6)$$

is formulated at very short distances, or, equivalently, at a very high normalization point $\mu = M_0$, where M_0 is the mass of an ultraviolet regulator. In other words, the normalization point is assumed to be much higher than all mass scales in the theory, in particular, $\mu \gg m_Q$. An effective theory for describing the low energy properties is obtained by evolving the Lagrangian from the high scale M_0 down to a lower normalization point μ . This means that we integrate out, step by step, all high frequency modes in the theory thus calculating the Lagrangian $\mathcal{L}(\mu)$. The latter is a full-fledged Lagrangian with respect to the soft modes with characteristic frequencies less than μ . The hard (high frequency) modes determine the coefficient functions

in $\mathcal{L}(\mu)$, while the contribution of the soft modes is hidden in the matrix elements of (an infinite set of) operators appearing in $\mathcal{L}(\mu)$. The value of this approach, outlined by Wilson long ago [1], has become widely recognized and exploited in countless applications. The peculiarity of the heavy quark theory lies in the fact that the *in* and *out* states contain heavy quarks. Although we integrate out the field fluctuations with the frequencies down to $\mu \ll m_Q$, the heavy quark fields themselves are not integrated out since we consider the sector with nonzero heavy flavor charge. The effective Lagrangian $\mathcal{L}(\mu)$ acts in this sector. Since the heavy quarks are neither produced nor annihilated, any sector with the given Q -number is treated separately from all others.

2 Basics of the Heavy Quark Theory

2.1 Effective Hamiltonian

The strategy for integrating out virtual degrees of freedom to pass on to an effective theory of heavy particles is described in the textbooks. The nonrelativistic fermion field $Q(x)$ has four degrees of freedom. Two of them, $\Psi(x)$ in Eq. (3) are nearly on-shell and two $\chi(x)$ are highly virtual describing excitation of antiquarks. One needs to integrate out first the antiparticle fields $\chi(x)$. For simplicity, we consider this in the rest frame.

Integrating out the antiparticle fields can be done straightforwardly since the QCD Lagrangian is bilinear in the quark fields. In this way one would obtain the (tree level version of the) so-called Lagrangian of HQET $\mathcal{L}_{\text{HQET}}$. It is a correct nonrelativistic Lagrangian up to the first order in $1/m_Q$.

This does not complete the program, however: one yet has a full ‘particle’ field $\Psi(x)$ which includes all frequencies from 0 to ∞ . Such a problem does not show up in the potential models where modes with $k \gg \Lambda_{\text{QCD}}$ are not excited. However, in actual QCD all radiative corrections would diverge due to large momentum gluons.

Therefore, besides the antiparticle fields one needs to integrate out hard gluons and the high frequency components of the heavy quark field $\Psi(x)$ itself, those for which $|\vec{k}|, \omega > \mu$. The scale μ is the normalization point of the effective theory. Since the configurations we integrate out depend on μ , the remaining effective low energy theory is also μ -dependent.

In practice, we want to have $\mu \ll m_b$ and, actually, as low as possible. On the other hand, μ must still belong to the perturbative domain. In practice this means that the best choice routinely adopted for applications is $\mu \sim \text{a few} \times \Lambda_{\text{QCD}}$, from 0.7 to 1 GeV. All coefficients in the effective Lagrangian obtained in this way are functions of μ .

To summarize, in treating heavy quarks we separate all strong interaction effects into ‘hard’ and ‘soft’ introducing a normalization scale μ . To calculate the effect of the short distance (perturbative) physics we use the original QCD Lagrangian Eq. (6). The soft physics is treated by the nonrelativistic Lagrangian where the

heavy quarks are represented by the corresponding nonrelativistic fields φ_Q :

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & -\frac{1}{4}G_{\mu\nu}^2 + \sum_q \bar{q}(i\not{D} - m_q)q + \sum_Q \left\{ -m_Q \varphi_Q^\dagger \varphi_Q + \varphi_Q^\dagger iD_0 \varphi_Q - \right. \\ & \left. - \frac{1}{2m_Q} \varphi_Q^\dagger (\vec{\sigma} i\vec{D})^2 \varphi_Q - \frac{1}{8m_Q^2} \varphi_Q^\dagger [-(\vec{D}\vec{E}) + \vec{\sigma} \cdot \{\vec{E} \times \vec{\pi} - \vec{\pi} \times \vec{E}\}] \varphi_Q \right\} \end{aligned} \quad (7)$$

where

$$\vec{\pi} \equiv i\vec{D} = \vec{p} - \vec{A}, \quad (\vec{\sigma} i\vec{D})^2 = (\vec{\sigma} \vec{\pi})^2 = \vec{\pi}^2 + \vec{\sigma} \vec{B}; \quad (8)$$

for simplicity only the tree level μ -independent coefficients are given. By the standard rules one constructs from the Lagrangian (7) the corresponding Hamiltonian. The heavy quark part takes the form³

$$\mathcal{H}_Q = -A_0 + \frac{1}{2m_Q} (\vec{\pi}^2 + \vec{\sigma} \vec{B}) + \frac{1}{8m_Q^2} [-(\vec{D}\vec{E}) + \vec{\sigma} \cdot \{\vec{E} \times \vec{\pi} - \vec{\pi} \times \vec{E}\}] + \mathcal{O}(1/m_Q^3). \quad (9)$$

The first term in the $1/m_Q^2$ part is called the Darwin term and the second one is the convection current (spin-orbital, or *LS*) interaction.

Since the external interactions (electromagnetic, weak *etc.*) are given in terms of the full QCD fields $Q(x)$, one needs also the relation between $Q(x)$ and $\varphi_Q(x)$:

$$\varphi_Q = \left(1 + \frac{(\vec{\sigma} \vec{\pi})^2}{8m_Q^2} + \dots \right) \frac{1+\gamma_0}{2} Q, \quad \frac{1-\gamma_0}{2} Q = \frac{\not{\pi}}{2m_Q} Q. \quad (10)$$

Let us briefly recall the textbook procedure for obtaining the nonrelativistic Lagrangian. One starts with

$$\mathcal{L}_{\text{heavy}}^0 = \bar{Q}(x)(i\not{D} - m_Q)Q(x) \quad (11)$$

and factors out of the $Q(x)$ field the “mechanical” time-dependent factor associated with the rest energy m_Q :

$$Q(x) = e^{-im_Q t} \tilde{Q}(x). \quad (12)$$

In an arbitrary frame moving with four-velocity v_μ it takes the following form:

$$Q(x) = e^{-im_Q v_\mu x_\mu} \tilde{Q}(x). \quad (13)$$

Then

$$iD_\mu Q(x) = e^{-im_Q(vx)} (m_Q v_\mu + \pi_\mu) \tilde{Q}(x), \quad \pi_\mu \equiv \hat{P}_\mu - m_Q v_\mu. \quad (14)$$

The Dirac equation $(i\not{D} - m_Q)Q = 0$ takes the form (now the tilde on Q is omitted)

$$\frac{1-\gamma_0}{2} Q = \frac{\not{\pi}}{2m_Q} Q, \quad \pi_0 Q = -\frac{\pi^2 + \frac{i}{2}\sigma G}{2m_Q} Q. \quad (15)$$

³The leading term A_0 is often omitted here. It is then implied that the time evolution operator is $\pi_0 = i\frac{\partial}{\partial t} + A_0$ rather than $i\frac{\partial}{\partial t}$ in the usual Schrödinger equation. This can be consistently carried out through the analysis.

$$\frac{i}{2}\sigma G = \frac{i}{2}\sigma_{\mu\nu}G^{\mu\nu}, \quad iG_{\mu\nu} = [\pi_\mu, \pi_\nu] = [\hat{P}_\mu, \hat{P}_\nu]$$

which allows one to exclude the small low components $\frac{1-\gamma_0}{2}Q(x)$ expressing them *via* the ‘large’ upper components $\frac{1+\gamma_0}{2}Q(x)$. For example, we have the useful identity

$$\bar{Q}Q = \bar{Q}\gamma_0 Q + 2\bar{Q}\left(\frac{1-\gamma_0}{2}\right)^2 Q = \bar{Q}\gamma_0 Q + \bar{Q}\frac{\pi^2 + \frac{i}{2}\sigma G}{2m_Q^2} Q + \text{total derivative} . \quad (16)$$

A subtlety emerges on this route that must be treated properly: at order $1/m_Q^2$ the time derivative ∂_0 appears with the nontrivial coefficient depending, for example, on the gluon field. This can be eliminated, and the time derivative returned to its canonical form performing the Foldy-Wouthuysen transformation

$$\varphi(x) = \left(1 + \frac{(\vec{\sigma}\vec{\pi})^2}{8m_Q^2} + \dots\right) \frac{1+\gamma_0}{2} Q(x) . \quad (17)$$

The illustration of the necessity for this field redefinition in the context of the heavy quark applications can be found in Ref. [2], Sect. 2.1.

2.2 Applications to spectroscopy of heavy flavor hadrons

To illustrate the consequences of the heavy quark Hamiltonian, let us consider the masses of hadrons containing a single heavy quark. This is not a dynamic question and requires only symmetry properties of \mathcal{H}_Q . It is described in much detail in a number of old reviews [3].

The mass of a hadron H_Q is given by the expectation value of the Hamiltonian:

$$M_{H_Q} = \frac{1}{2M_{H_Q}} \langle H_Q | \mathcal{H}_{\text{tot}} | H_Q \rangle , \quad (18)$$

and we have

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{light}} + \mathcal{H}_Q + m_Q ,$$

where we can expand

$$\mathcal{H}_Q = \mathcal{H}_0 + \frac{1}{m_Q}\mathcal{H}_1 + \frac{1}{m_Q^2}\mathcal{H}_2 + \dots \quad (19)$$

with

$$\begin{aligned} \mathcal{H}_0 &= -\varphi_Q^+ A_0 \varphi_Q \xrightarrow{\text{QM}} -A_0(0) [\{x_{\text{light}}\}] , \\ \frac{1}{m_Q}\mathcal{H}_1 &= \frac{1}{2m_Q} (\vec{\pi}^2 + \vec{\sigma}\vec{B}) , \\ \frac{1}{m_Q^2}\mathcal{H}_2 &= \frac{1}{8m_Q^2} [-(\vec{D}\vec{E}) + \vec{\sigma} \cdot \{\vec{E} \times \vec{\pi} - \vec{\pi} \times \vec{E}\}] , \end{aligned} \quad (20)$$

etc. Therefore,

$$M_{H_Q} = m_Q + \bar{\Lambda} + \frac{1}{2m_Q} \frac{\langle H_Q | \vec{\pi}^2 + \vec{\sigma} \vec{B} | H_Q \rangle}{2M_{H_Q}} + \dots = m_Q + \bar{\Lambda} + \frac{(\mu_\pi^2 - \mu_G^2)_{H_Q}}{2m_Q} + \dots \quad (21)$$

The mass expansion was alternatively derived in Ref. [4] using the formalism based on the trace of the energy-momentum tensor.

In Eq. (21) we introduced the notations μ_π^2 , μ_G^2 for the expectation values of two $D=5$ heavy quark operators which will often appear in our discussion:

$$\begin{aligned} \mu_\pi^2 &= \langle H_Q | \vec{\pi}^2 | H_Q \rangle_{\text{QM}} \equiv \frac{1}{2M_{H_Q}} \langle H_Q | \bar{Q} \vec{\pi}^2 Q(0) | H_Q \rangle_{\text{QFT}} \\ \mu_G^2 &= -\langle H_Q | \vec{\sigma} \vec{B} | H_Q \rangle_{\text{QM}} \equiv \frac{1}{2M_{H_Q}} \langle H_Q | \bar{Q} \frac{i}{2} \sigma_{\mu\nu} G^{\mu\nu} Q(0) | H_Q \rangle_{\text{QFT}} . \end{aligned} \quad (22)$$

The physical meaning of μ_π^2 is quite evident: the heavy quark inside H_Q experiences a *zitterbewegung* due to its coupling to light cloud. Its average spatial momentum squared is μ_π^2 . The second expectation value measures the amount of the chromo-magnetic field produced by the light cloud at the position of the heavy quark. In principle, the actual heavy hadron states H_Q depend on m_Q via the $1/m_Q$ -suppressed terms of the Hamiltonian. Therefore, the above expectation values also have such terms. Often it is convenient to consider the asymptotic values at $m_Q \rightarrow \infty$, and to use, correspondingly, the eigenstates of the $m_Q \rightarrow \infty$ Hamiltonian $\mathcal{H}_0 + \mathcal{H}_{\text{light}}$.

The parameter $\bar{\Lambda}$ appearing in Eq. (21) was introduced as a constant in the Heavy Quark Effective Theory (HQET [5]) in [6]; it is associated with those terms in the effective Lagrangian $\mathcal{L}(\mu)$ (disregarded so far) which are entirely due to the light degrees of freedom. Needless to say that in the Wilsonian approach $\bar{\Lambda}$ is actually μ dependent, $\bar{\Lambda}(\mu)$. Wherever there is the menace of confusion the μ dependence of $\bar{\Lambda}$ will be indicated explicitly.

There exists an interesting expression for this parameter through the anomaly in the trace of the energy-momentum tensor, derived in Ref. [4]:

$$\bar{\Lambda} = \frac{1}{2M_{H_Q}} \langle H_Q | \frac{\beta(\alpha_s)}{4\alpha_s} G^2 + \sum_q m_q (1 + \gamma_m) \bar{q} q | H_Q \rangle_{m_Q \rightarrow \infty} . \quad (23)$$

Here γ_m is the anomalous dimension of the light quark mass. This equation parallels the similar relation for the nucleon mass [7] well known in the chiral limit when all quark masses are neglected:

$$M_N = \frac{1}{2M_N} \langle N | \frac{\beta(\alpha_s)}{4\alpha_s} G^2 | N \rangle .$$

It should be noted, however, that the operator G^2 in Eq. (23), although local and gauge invariant, is not a *local heavy-quark* operator as they are understood in the framework of the heavy quark expansion: the latter must be of the form $\bar{Q} \dots Q$

where ellipses denote a local operator at the same space-time point as the heavy quark fields Q . Quantum mechanically, the operators in Eq. (23) describe the space integral over the whole volume occupied by the heavy hadron, rather than the value of the gluonic and quark fields at the position of the heavy quark.

The renormalization properties of the operator G^2 are quite different in the sector of QCD with the heavy quark. An accurate consideration reveals [4] that an additional term $-\mu \frac{dm_Q}{d\mu}$ must be added in the r.h.s. of Eq. (23), where μ stands for the normalization point. The normalization point dependence of the heavy quark parameters will be discussed in more detail in the subsequent sections.

The value of $\bar{\Lambda} = \lim_{m_Q \rightarrow \infty} (M_{H_Q} - m_Q)$ has the scale of Λ_{QCD} and depends on the state of light degrees of freedom. These states generally carry spin j . In the limit $m_Q \rightarrow \infty$ the heavy quark spin decouples since the spin-dependent parts are present only starting \mathcal{H}_1 . Thus, the heavy flavor hadrons can be classified not only by their total spin J but by the spin of light degrees of freedom j . It would be just the overall spin of a hadron in the hypothetical world with the spinless heavy quarks discussed in Sect. 1.1. Unless $j = 0$, there are two values of the total spin $J = j \pm \frac{1}{2}$. The corresponding states form ‘hyperfine’ multiplets and are degenerate up to $1/m_Q$ corrections. They are, for example

$$j = 1/2 \quad \left\{ \begin{array}{lll} D, & B & J = 0 \\ D^*, & B^* & J = 1 \end{array} \right. , \quad j = 0, J = 1/2 \quad \Lambda_c, \Lambda_b$$

For Λ_Q -baryons all spin is carried by the heavy quark (up to $1/m_Q$ corrections). The observed spectroscopy of these states clearly supports this picture:

$$\begin{aligned} M_{\Lambda_b} - M_B &\simeq 350 \text{ MeV} & M_{B_s} - M_{B^-} &\simeq 90 \text{ MeV} \\ M_{\Lambda_c} - M_D &\simeq 420 \text{ MeV} & M_{D_s} - M_{D^0} &\simeq 104 \text{ MeV} \end{aligned} \quad (24)$$

These relations are easily improved including $1/m_Q$ terms, Eq. (21). The operator $\bar{Q}\vec{\pi}^2 Q$ is spin-independent and its expectation value is the same for all members of a hyperfine multiplet. It does not split masses inside the multiplet.

The chromomagnetic operator $\bar{Q}\frac{i}{2}\sigma G Q = -2\vec{S}_Q\vec{B}(0)$ depends on the heavy quark spin \vec{S}_Q and thus lifts degeneracy leading to the hyperfine splitting among the members of the multiplet. Restricted to a particular hyperfine multiplet, the chromomagnetic field $\vec{B}(0)$ is proportional to the spin of light degrees of freedom: $\vec{B} = c \cdot \vec{j}$. Therefore,

$$\langle \vec{\sigma} \vec{B} \rangle = 2c \langle \vec{S}_Q \vec{j} \rangle = c \left(J(J+1) - j(j+1) - \frac{3}{4} \right). \quad (25)$$

It is easy to see that

$$\sum_{H_Q} \langle H_Q | \bar{Q} \vec{\sigma} \vec{B} Q | H_Q \rangle \equiv \text{Tr} \bar{Q} \vec{\sigma} \vec{B} Q = 0 \quad (26)$$

always holds if the summation is performed over a hyperfine multiplet. Therefore, for example,

$$\mu_G^2(B) + 3\mu_G^2(B^*) = 0. \quad (27)$$

In the Λ_b family the expectation value of $\bar{b}_2^i \sigma G b$ vanishes.

So far most of the practical applications refer to B mesons; as a result, usually μ_π^2 proper denotes the expectation value of the kinetic operator just in B or B^* . Likewise, $\mu_G^2 \equiv \mu_G^2(B) = -3\mu_G^2(B^*)$. Experimentally,

$$M_{B^*} - M_B \simeq \left(\frac{1}{3} + 1\right) \frac{\mu_G^2}{2m_b} = \frac{4}{3} \frac{\mu_G^2}{2m_b} \simeq 46 \text{ MeV} . \quad (28)$$

Neglecting the difference between $M_B + M_{B^*}$ and $2m_b$ (which is formally an effect of higher order in $1/m_b$), one can write

$$\mu_G^2 \simeq \frac{4}{3} (M_{B^*}^2 - M_B^2) \simeq 0.36 \text{ GeV}^2 . \quad (29)$$

In charmed mesons $M_{D^*} - M_D \simeq 140 \text{ MeV}$ which agrees with the fact that this hyperfine splitting is proportional to $1/m_c$. The hadron mass averaged over a hyperfine multiplet, *e.g.* $\overline{M}_B = \frac{3M_{B^*} + M_B}{4}$ is affected at order $1/m_Q$ by only the kinetic energy term $\mu_\pi^2/2m_b$.

The mass expansion can be extended to higher orders in $1/m_Q$. For example, to order $1/m_Q^2$ the hadron mass takes the form [4]

$$\begin{aligned} M_{H_Q} = m_Q + \overline{\Lambda} + \frac{1}{m_Q} \left\{ \frac{1}{2M_{H_Q}} \langle H_Q | \bar{Q} \frac{(\vec{\sigma} \vec{\pi})^2}{2} Q | H_Q \rangle \right\}_{m_Q=\infty} + \\ + \frac{1}{8m_Q^2} \left\{ \frac{1}{2M_{H_Q}} \langle H_Q | \bar{Q} \left(-(\vec{D} \vec{E}) + 2\vec{\sigma} \cdot \vec{E} \times \vec{\pi} \right) Q | H_Q \rangle \right\}_{m_Q=\infty} - \frac{\rho^3}{4m_Q^2} + \mathcal{O}(m_Q^{-3}) . \end{aligned} \quad (30)$$

The nonlocal (positive) correlator ρ^3 is the second-order iteration of the $1/m_Q$ part of the Hamiltonian:

$$\rho^3 = i \int d^4x \frac{1}{4M_{H_Q}} \langle H_Q | T \{ \bar{Q} (\vec{\sigma} \vec{\pi})^2 Q(x), \bar{Q} (\vec{\sigma} \vec{\pi})^2 Q(0) \} | H_Q \rangle'_{m_Q=\infty} \quad (31)$$

(the prime indicates that the diagonal transitions must be removed from the correlation function). In B mesons the expectation values of the two terms in $1/m_Q^2$ Hamiltonian, the Darwin and the convection current interactions, are denoted as

$$\begin{aligned} \rho_D^3 &= \frac{1}{2M_B} \langle B | \bar{b} \left(-\frac{1}{2} \vec{D} \vec{E} \right) b | B \rangle = \frac{1}{2M_{B^*}} \langle B^* | \bar{b} \left(-\frac{1}{2} \vec{D} \vec{E} \right) b | B^* \rangle \\ \rho_{LS}^3 &= \frac{1}{2M_B} \langle B | \bar{b} \vec{\sigma} \cdot \vec{E} \times \vec{\pi} b | B \rangle = -\frac{3}{2M_{B^*}} \langle B^* | \bar{b} \vec{\sigma} \cdot \vec{E} \times \vec{\pi} b | B^* \rangle . \end{aligned} \quad (32)$$

The Darwin operator by virtue of QCD equations of motions equals to the local four-fermion operator of the form $-\frac{g_s^2}{2} \bar{Q} \frac{\lambda^a}{2} Q \sum_q \bar{q} \gamma_0 \frac{\lambda^a}{2} q$.

A similar decomposition can be made for the nonlocal correlators whose expectation values were generically denoted by ρ^3 . In particular, in pseudoscalar and vector ground states

$$\rho_{\pi\pi}^3 = i \int d^4x \frac{1}{4M_B} \langle B | T \{ \bar{b} \pi^2 b(x), \bar{b} \pi^2 b(0) \} | B \rangle'$$

$$\rho_{\pi G}^3 = i \int d^4x \frac{1}{2M_B} \langle B | T \{ \bar{b} \vec{\pi}^2 b(x), \bar{b} \vec{\sigma} \vec{B} b(0) \} | B \rangle'$$

$$\frac{1}{3} \rho_S^3 \delta_{ij} \delta_{kl} + \frac{1}{6} \rho_A^3 (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) = i \int d^4x \frac{1}{4M_B} \langle B | T \{ \bar{b} \sigma_i B_k b(x), \bar{b} \sigma_j B_l b(0) \} | B \rangle' . \quad (33)$$

Then one has for ρ^3 in B and B^* , respectively,

$$(\rho^3)_B = \rho_{\pi\pi}^3 + \rho_{\pi G}^3 + \rho_S^3 + \rho_A^3 \quad , \quad (\rho^3)_{B^*} = \rho_{\pi\pi}^3 - \frac{1}{3} \rho_{\pi G}^3 + \rho_S^3 - \frac{1}{3} \rho_A^3 . \quad (34)$$

These parameters also give the $1/m_Q$ corrections to the expectation values of the kinetic and chromomagnetic operators [4].

In many applications one needs to know the difference between m_b and m_c . This difference is well constrained in the heavy quark expansion. For example,

$$m_b - m_c = \frac{M_B + 3M_{B^*}}{4} - \frac{M_D + 3M_{D^*}}{4} + \frac{\mu_\pi^2}{2} \left(\frac{1}{m_c} - \frac{1}{m_b} \right) + \frac{\rho_D^3 - \bar{\rho}^3}{4} \left(\frac{1}{m_c^2} - \frac{1}{m_b^2} \right) + \mathcal{O}\left(\frac{1}{m^3}\right). \quad (35)$$

Here μ_π^2 is the asymptotic expectation value of the kinetic operator and $\bar{\rho}^3 \equiv \rho_{\pi\pi}^3 + \rho_S^3$ is the sum of two positive nonlocal correlators. As will be discussed in the subsequent sections, all quantities in Eq. (35) but the meson masses depend on the normalization point which can be arbitrary, except that it must be much lower than $m_{c,b}$. Evaluating the above expansion we arrive at

$$m_b - m_c \simeq 3.50 \text{ GeV} + 40 \text{ MeV} \frac{\mu_\pi^2 - 0.5 \text{ GeV}^2}{0.1 \text{ GeV}^2} + \Delta M_2 , \quad |\Delta M_2| \lesssim 0.02 \text{ GeV} , \quad (36)$$

where plausible assumptions about the $D=6$ expectation values have been made. The values of the hadronic parameters will be discussed later. Let us mention that the $m_b - m_c$ estimate at $\mu_\pi^2 \approx 0.5 \text{ GeV}^2$ appears to be in a good agreement with the separate determinations of m_b and m_c from the sum rules in charmonia and Υ .

2.3 Heavy quark symmetry for formfactors

The amplitudes of the semileptonic weak $b \rightarrow c$ decays are described by the corresponding transition formfactors. Typical semileptonic $b \rightarrow c$ decays as they look like in Feynman diagrams are shown in Figs. 1. The hadronic part of the weak decay Hamiltonian mediating such decays is

$$\mathcal{H}_{\text{weak}} = \int d^3x e^{-i\vec{q}\vec{x}} \bar{c} \gamma_\mu (1 - \gamma_5) b(x) . \quad (37)$$

It says that b with a momentum \vec{p} is instantaneously replaced by the c quark with the momentum $\vec{p} - \vec{q}$. The resulting state hadronizes into eigenstates of the Hamiltonian corresponding to $m_Q = m_c$, that is, must be projected onto such states.

The space-time picture of the decay is simple for heavy quarks. At $t < 0$ the initial b hadron is at rest and constitutes a coherent state of light degrees of freedom

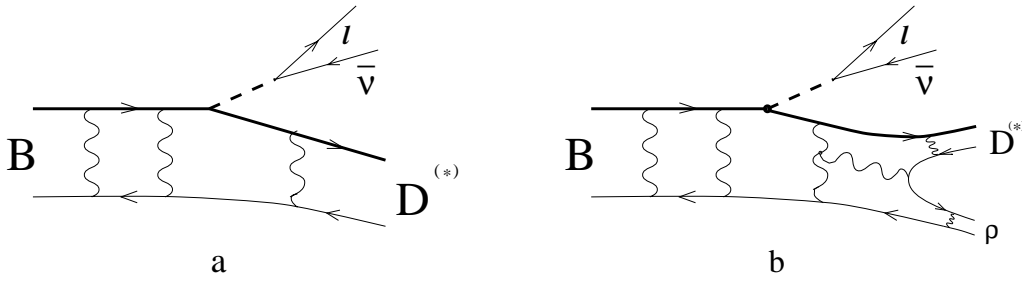


Figure 1: Quark diagrams for the exclusive $B \rightarrow D^{(*)}$ (a) and generic (b) semileptonic decays.

in the static field of the heavy quark. At $t=0$ the b quark emits the lepton pair with the momentum \vec{q} and transforms into a c quark. The c quark gets the recoil momentum $-\vec{q}$ and starts moving with the velocity $\vec{v} = -\vec{q}/m_c$. Such a state is not anymore an eigenstate of the Hamiltonian, and afterwards undergoes nontrivial evolution. The light cloud can get a coherent boost along the direction of $-\vec{q}$ and form again the same ground-state, or excited hadron. Alternatively, it can crack apart and produce a few-body final hadronic state.

The heavy quark symmetry *per se* cannot help calculating the amplitudes to create such final states. However, it tells one that the hadronization process does not depend on the heavy quark spin, or on the concrete value of the mass m_Q but rather on the velocity of the final state heavy hadron. This velocity cannot change in the process of hadronization if only soft gluons are exchanged between the heavy quark and the light cloud. This independence holds, of course, only when m_Q is very large (and the final state quark does not move too fast); there are various $1/m_Q$ corrections at finite masses.

Let us consider, for example, the ground state transition $B \rightarrow D$. Its amplitude depends on the velocity \vec{v} of D , $f(\vec{v}^2)$. The very same function would describe also decays $B \rightarrow D^*$, or the elastic amplitude of scattering of a photon on the b quark $B \rightarrow B$, $B \rightarrow B^*$. Moreover, in the proper normalization $f(0) = 1$ holds. This fact follows from the conservation of the b -quark vector current, for the amplitude at zero momentum transfer measures the total ‘ b -quark charge’ of the hadron $n_b - n_{\bar{b}}$. Its origin is simply understood: if $\vec{v}=0$, the final state is not really disturbed by movement of the static source. A nontrivial rearrangement of the light cloud for $B \rightarrow D$ is associated only with the mass-dependent terms in \mathcal{H}_Q vanishing when $m_Q \rightarrow \infty$.

Consider the vector $\bar{b}\gamma_\mu b$ current in B meson. It is described by the single formfactor $f_+(q^2)$:

$$\langle B(p') | \bar{b}\gamma_\mu b(0) | B(p) \rangle = f_+(q^2) (p + p')_\mu, \quad q_\mu = p_\mu - p'_\mu \quad (38)$$

(the second structure $(p+p')_\mu$ is forbidden by T invariance or current conservation). The value $f_+(0)$ measures the total ‘beauty charge’ of the hadron and is not renor-

malized by the strong interaction, $f_+(0) = 1$.⁴ Passing to the velocities, we use instead of q^2 the scalar product $v_\mu v'_\mu$:

$$(vv') = \frac{(pp')}{M_B^2} = 1 - \frac{(q^2)}{2M_B^2} \geq 1, \quad (39)$$

and

$$f_+(q^2) = \xi(vv'), \quad \xi(1) = 1. \quad (40)$$

$\xi(vv')$ is called the Isgur-Wise function. The heavy quark symmetry then states that

$$\langle D(v') | \bar{c} \gamma_\mu b(0) | B(v) \rangle = \left(\frac{M_B + M_D}{2\sqrt{M_B M_D}} (p + p')_\mu - \frac{M_B - M_D}{2\sqrt{M_B M_D}} (p - p')_\mu \right) \xi(vv'). \quad (41)$$

D^* differs from D only by the alignment of the c quark spin. Taking this into account yields

$$\begin{aligned} \langle D^*(v', \epsilon) | \bar{c} \gamma_\mu b(0) | B(v) \rangle &= i \epsilon_{\mu\nu\alpha\beta} \epsilon_\nu^* v'_\alpha v_\beta \xi(vv') \sqrt{M_B M_D^*} \\ \langle D^*(v', \epsilon) | \bar{c} \gamma_\mu \gamma_5 b(0) | B(v) \rangle &= \left\{ \epsilon_\mu^* (vv' + 1) - v'_\mu (\epsilon^* v) \right\} \xi(vv') \sqrt{M_B M_D^*}. \end{aligned} \quad (42)$$

It is important that these relations are valid in the limit $m_{b,c} \rightarrow \infty$ and if no short-distance radiative corrections were present. The corrections to the symmetry limit are minimal at $\vec{q} = 0$ ($v = v'$, the so-called zero recoil point); numerically the value of the axial formfactor was estimated to be $F_{B \rightarrow D^*}(0) \simeq 0.9$ [9]. The corrections at arbitrary $\vec{v} \sim 1$ are generally significant.

2.4 Feynman rules at $m_Q \rightarrow \infty$

The Feynman rules for heavy quarks are usual propagators and vertices for nonrelativistic particles where $1/m \rightarrow 0$. If the heavy quark momentum is $p_\mu = (m_Q + \omega, \vec{p})$ then

$$\begin{aligned} G(p) &= \lim_{m \rightarrow \infty} \delta_{\alpha\beta} \frac{1}{\omega - i\epsilon - \frac{\vec{p}^2}{2m}} \delta_{ij} = \frac{\delta_{\alpha\beta}}{\omega - i\epsilon} \delta_{ij} \\ \Gamma_\mu &= g_s \frac{\lambda_{\alpha\beta}^a}{2} \delta_{\mu 0}. \end{aligned} \quad (43)$$

Here a, α, β are color indices and spinor indices i, j take values 1 or 2. (It is often advantageous to keep the nonrelativistic term $\vec{p}^2/2m_Q$ in the propagator as an infrared regulator.) These rules follow immediately from the static Lagrangian

$$\mathcal{L}_Q = \varphi_Q^\dagger i D_0 \varphi_Q, \quad D_0 = \partial_0 - i A_0^a \frac{\lambda^a}{2}.$$

The same nonrelativistic system can be considered in an arbitrary moving frame, where one can write

$$\mathcal{L}_v = \varphi_v^\dagger i(vD) \varphi_v. \quad (44)$$

⁴For simplicity, we use the convention that B consists of b and \bar{q} . That is, B^- is a B meson while B^+ is \bar{B} .

Instead of the nonrelativistic spinor φ_Q one then considers the “bispinor” $\varphi_v(x) = \frac{1+\not{v}}{2}\tilde{Q}(x)$; the propagator is written as

$$\frac{1+\not{v}}{2} \frac{m_Q + \not{p} - \not{k}}{m_Q^2 - (p-k)^2 - i\epsilon} \frac{1+\not{v}}{2} \longrightarrow \frac{1+\not{v}}{2} \frac{1}{vk - i\epsilon} \quad (45)$$

$$\Gamma_\mu = g_s \frac{\lambda^a}{2} v_\mu ,$$

where $k = p - m_Q v$. Of course, such a generalization can be useful only if the initial and final state hadrons have different velocities, $\vec{v} \neq \vec{v}'$. In that case the external (‘weak’) source carries a large momentum $\vec{q} \sim m_Q \delta \vec{v}$.

2.4.1 Subtleties of actual QCD

It must be noted, however, that the field-theoretic description of processes where the heavy quarks change velocity becomes subtle if the quantum radiative corrections are really incorporated. Any change in the velocity of a static source leads to actual radiation of real hadrons with momenta \vec{k} all the way up to m_Q :

$$\frac{dw}{d\omega} \sim \frac{\alpha_s(\omega)}{\omega} (\vec{v}' - \vec{v})^2 ; \quad (46)$$

here ω denotes the radiated energy. This means that with infinitely heavy quarks one would actually observe radiation of light hadrons off the heavy flavor hadrons with arbitrary large energies. Without an ultraviolet cutoff any exclusive transition probability would vanish being suppressed by the more or less universal factor (the square of the non-Abelian analogue of the nonrelativistic Sudakov formfactor)

$$S \sim e^{-\frac{4\alpha_s}{9\pi}(\Delta\vec{v})^2 \ln \frac{\mu}{\epsilon}} . \quad (47)$$

Here μ denotes the ultraviolet cutoff and ϵ determines the energy resolution for the final hadronic state (for the validity of the perturbative expansion ϵ must be much large than Λ_{QCD} ; taking it around the typical hadronic scale gives the estimate of the overall perturbative suppression of the exclusive transition probability). This factor has a meaning of the probability of the heavy color source *not to emit* gluons in the interval of energies between ϵ and μ in the act of acceleration.

Beyond the small velocity approximation the factor $(\Delta\vec{v})^2$ in Eq. (47) must be replaced by $\frac{3}{2} \left(\frac{1}{|\Delta\vec{v}|} \ln \frac{1+|\Delta\vec{v}|}{1-|\Delta\vec{v}|} - 2 \right)$ which can be read off the expression for the intensity of electromagnetic radiation in classical electrodynamics:

$$\frac{1}{\omega} \frac{dI(\omega)}{d\omega} = \frac{\alpha}{\pi} \left(\frac{1}{|\Delta\vec{v}|} \ln \frac{1+|\Delta\vec{v}|}{1-|\Delta\vec{v}|} - 2 \right) \frac{1}{\omega} . \quad (48)$$

It is remarkable that this law is not modified by quantum corrections [8] as long as ω remains much smaller than the masses of all charged particles and the effects of their virtual production can be neglected.

In the non-Abelian theories like QCD the situation is different, even in the dipole approximation $|\vec{v}| \ll 1$. The non-Abelian QCD dipole radiation was considered in detail in Ref. [10]. In contrast with QED where the coupling for the soft photon radiation is exactly $\alpha_{\text{em}}(0)$ and is not renormalized by quantum corrections, in QCD the dipole radiation is governed by the effective coupling $\alpha_s^{(d)}(\omega)$ which runs with the energy scale:

$$\frac{\alpha_s^{(d)}(\omega)}{\pi} = \frac{\bar{\alpha}_s(e^{-5/3+\ln^2\omega})}{\pi} - N_c \left(\frac{\pi^2}{6} - \frac{13}{12} \right) \left(\frac{\alpha_s}{\pi} \right)^2 + \mathcal{O}(\alpha_s^3), \quad (49)$$

with $\bar{\alpha}_s$ the $\overline{\text{MS}}$ coupling. It can be shown that the second (conformal) term in the $\mathcal{O}(\alpha_s^2)$ part of the coupling must equal twice the corresponding part of the so-called cusp anomalous dimension of the Wilson lines (at small angle) investigated in detail in the mid 80's [11]. The dipole radiation coupling governs the normalization point evolution of the heavy quark masses and a number of heavy quark operators in the effective low-energy theory.

As any effective coupling, $\alpha_s^{(d)}(\omega)$ is not a purely perturbative object at arbitrary precision. It has a power-suppressed nonperturbative component at arbitrarily large energy ω which, for example, depends on the particular type of the heavy flavor hadron which is accelerated. (Pure perturbation theory does not depend on the light cloud surrounding heavy quark for actual hadrons.) Using the OPE approach it was shown [10] that such nonperturbative effects in the dipole radiation fade out at least as $1/\omega^3$, and for the ground state hadrons (B mesons) it was estimated that

$$\left(\delta\alpha_s^{(d)}(\omega) \right)_{\text{nonpert}} \approx - \left(\frac{0.6 \text{ GeV}}{\omega} \right)^3. \quad (50)$$

In general, the perturbative factors suppressing the velocity-changing transition amplitudes of infinitely heavy quarks are universal and related to the cusp anomalous dimensions of Wilson lines describing propagation of massive color objects. They appear in the renormalization of bent Wilson lines; the cusps correspond to instant changes in velocity and thus lead to new ultraviolet divergences. The dedicated discussion can be found in the original publications [11]. The nonperturbative aspects of factorization in the radiative effects have not been carefully studied yet.

3 Basic Parameters of the Heavy Quark Expansion

3.1 The heavy quark mass

3.1.1 What is m_Q ?

In quantum field theory the object we begin our work with is the Lagrangian formulated at some high scale M_0 . The mass m_0 is a parameter in this Lagrangian;

it enters on the same footing as, say, the bare coupling constant $\alpha_s^{(0)}$ with the only difference being that it carries dimension. As with any other coupling, m_0 enters all observable quantities in a certain combination with the ultraviolet cutoff M_0 , which is universal for a renormalizable theory.

The mass parameter m_0 by itself is not observable, like $\alpha_s^{(0)}$. For calculating observable quantities at the scale $\mu \ll M_0$ it is usually convenient to relate m_0 to some mass parameter relevant to the scale μ . Integrating out momentum scales above μ converts $\alpha_s^{(0)}$ into $\alpha_s(\mu)$ – and likewise m_0 into $m(\mu)$. Such $m(\mu)$ is not something absolute since depends on μ . It is either used on the same footing as $\alpha_s(\mu)$ or, in the final expressions, is eliminated in favor of some suitable observable mass. For example, in quantum electrodynamics (QED) at low energies (*i.e.* $E \ll m_e$) there is an obvious “best” candidate: the actual mass of an isolated electron, m_e . In the perturbative calculations it is determined as the position of the pole in the electron Green function (more exactly, the beginning of the cut). The advantages are evident: m_e is gauge-invariant and experimentally measurable.

The analogous parameter for heavy quarks in QCD is referred to as the pole quark mass, the position of the pole of the quark Green function. Like m_e it is gauge invariant. Unlike QED, however, the quarks do not exist as isolated objects (there are no states with the quark quantum numbers in the observable spectrum, and the quark Green function beyond a given order has neither a pole nor a cut). Hence, m^{pole} cannot be directly measured; m^{pole} exists only as a theoretical construction.

In principle, there is nothing wrong with using m^{pole} in *perturbation theory* where it naturally appears in the Feynman graphs for the quark Green functions, scattering amplitudes and so on. It may or may not be convenient, depending on concrete goals.

The pole mass in QCD is perturbatively infrared stable, order by order, like in QED (the formal proof was given recently in [12]). It is well-defined to every given order in perturbation theory. One cannot define it to all orders, however; the sum of the series does not converge to a definite number. In a sense, the pole mass is *not* infrared-stable nonperturbatively. Intuitively this is clear: since the quarks are confined in the full theory, the best one can do is to define the would-be pole position with an intrinsic uncertainty of order Λ_{QCD} [13].

Based on experience in QED or ordinary QM, non-existence of m_Q^{pole} may seem counter-intuitive. Employing perturbation theory, we start with the free quark propagator

$$G(p) = \frac{1}{m_Q(\mu) - \not{p}}$$

($m_Q(\mu)$ is the parameter entering the Lagrangian) which has a pole at $p^2 = m_Q^2(\mu)$ and, therefore, describes a particle with mass $m_Q(\mu)$. Accounting for the gluon exchanges to the first order in α_s adds the diagram Fig. 2 and the pole moves to $p^2 \simeq \left(m_Q(\mu) + \frac{4\alpha_s}{3\pi}\mu\right)^2$ describing now a particle with the mass $m_Q^{(1)} \simeq m_Q(\mu) + \frac{4\alpha_s}{3\pi}\mu$, *etc.* In any order of perturbation theory we see a quark pole and the corresponding particle with certain mass, differing from the mass in the Lagrangian. This mass is just the “pole” mass. Clearly, it depends on the order of perturbation theory one

considers, and on a concrete version of the employed expansion:

$$m_Q^{(k)} = m_Q(\mu) \sum_{n=0}^k C_n \left(\frac{\mu}{m} \right) \left(\frac{\alpha_s(\mu)}{\pi} \right)^n, \quad C_0 = 1. \quad (51)$$

It is tempting to define the ‘actual’ pole mass as a sum of the series (51). It appears, however, that the sum does not converge to a reasonable number and cannot be defined with the necessary accuracy in a motivated way, but suffers from an irreducible uncertainty of order Λ_{QCD} .

Before explaining the origin of this perturbative uncertainty, let us remark that such a definition of the quark mass is nothing but equating it with the lowest eigenvalue of the (perturbative) Hamiltonian in the sector with the number of heavy quarks 1. The eigenvalues of the Hamiltonian, however, are *not* short-distance quantities, and their calculation in the perturbative expansion is not adequate. This is in the nice correspondence with the observed fact that no isolated heavy quark exists in the spectrum of hadrons which contains instead B , B^{**} , Λ_b , $B_s \dots$ with masses differing by amount $\mathcal{O}(\Lambda_{\text{QCD}}^1)$.

The physical reason behind the perturbative instability of the long-distance regime is the growing of the interaction strength α_s . One can illustrate this instability in the following way. Consider the energy stored in the chromoelectric field in a sphere of radius $R \gg 1/m_Q$ around a static color source,

$$\delta \mathcal{E}_{\text{Coul}}(R) \propto \int_{1/m_b \leq |x| < R} d^3x \vec{E}_{\text{Coul}}^2 \propto \text{const} - \frac{\alpha_s(R)}{\pi} \frac{1}{R}. \quad (52)$$

This energy is what one adds to the bare mass of a heavy particle to determine what will be the mass including QCD interactions. The definition of the pole mass amounts to setting $R \rightarrow \infty$; *i.e.*, in evaluating the pole mass one undertakes to integrate the energy density associated with the color source over *all* space assuming that it has the Coulomb form. In real life the color interaction becomes strong at $R_0 \sim 1/\Lambda_{\text{QCD}}$; at such distances the chromoelectric field has nothing to do with the Coulomb tail. Thus, one cannot include the region beyond R_0 in a meaningful way. Its contribution which is of order Λ_{QCD} , thus, has to be considered as an irreducible uncertainty which is power-suppressed relative to m_Q ,

$$\frac{\delta_{\text{IR}} m_Q^{\text{pole}}}{m_Q} = \mathcal{O} \left(\frac{\Lambda_{\text{QCD}}}{m_Q} \right). \quad (53)$$

Exactly this behavior is traced formally in perturbation theory. In the nonrelativistic regime where the internal momentum $|\vec{k}| \ll m_Q$ the expression for the diagram Fig. 2 is simple,

$$\delta m_Q \sim -\frac{4}{3} \int \frac{d^4k}{(2\pi)^4 i k_0} \frac{4\pi\alpha_s}{k^2} = \frac{4}{3} \int \frac{d^3\vec{k}}{4\pi^2} \frac{\alpha_s}{\vec{k}^2}. \quad (54)$$

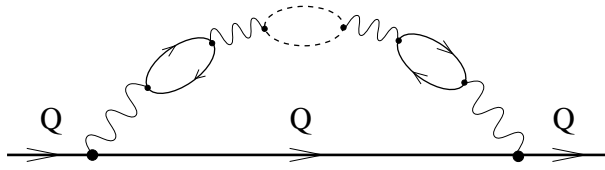


Figure 2: Perturbative diagrams leading to renormalization of the heavy quark mass. The contribution of the gluon momenta below m_Q expresses the classical Coulomb self-energy of the colored particle. The number of bubble insertions into the gluon propagator can be arbitrary generating corrections in all orders in α_s . The factorial growth of the coefficients produces the IR renormalon uncertainty in m_Q^{pole} of order Λ_{QCD} .

The latter expression is $\frac{1}{2}V(0)$ with $V(R)$ the usual Coulomb potential between two quarks. The running of the coupling is generated by dressing the gluon propagator by virtual pairs and leads to

$$\delta m_Q \simeq \frac{4}{3} \int \frac{d^3 \vec{k}}{4\pi^2} \frac{\alpha_s(\vec{k}^2)}{\vec{k}^2}. \quad (55)$$

Since

$$\alpha_s(k^2) = \alpha_s(\mu^2) \left\{ 1 + \frac{\alpha_s(\mu^2)}{4\pi} b \ln \frac{k^2}{\mu^2} \right\}^{-1}, \quad b = \frac{11}{3}N_c - \frac{2}{3}n_f, \quad (56)$$

we can expand $\alpha_s(k^2)$ in a power series in $\alpha_s(\mu^2)$ and easily find the $(n+1)$ -th order contribution to δm_Q ,

$$\frac{\delta m_Q^{(n+1)}}{m_Q} \sim \frac{4}{3} \frac{\alpha_s(\mu)}{\pi} n! \left(\frac{b\alpha_s(\mu)}{2\pi} \right)^n. \quad (57)$$

The coefficients grow factorially and contribute with the same sign. Therefore, one cannot define the sum of these contributions even using the trick with the Borel transformation. The best one can do is to truncate the series judiciously. An optimal truncation leaves us with an irreducible uncertainty $\sim \mathcal{O}(\Lambda_{\text{QCD}})$ [14, 15]. The above perturbative corrections are example of the so-called infrared renormalons [16, 17].

This uncertainty can be quantified. A formal Borel resummation of such non-summable series leads to the result which literally has an imaginary part which can be taken as a measure of the uncertainty. The imaginary part for the series Eq. (57) is

$$\text{Im } m_Q^{\text{pole}} = \frac{8\pi}{3b} e^{5/6} \Lambda_{\text{QCD}}^{\overline{\text{MS}}}. \quad (58)$$

It became conventional to assign the formal imaginary part divided by π to the irreducible uncertainty. Even with this minimal choice

$$\delta m_Q^{\text{pole}} = \frac{1}{\pi} |\text{Im } m_Q^{\text{pole}}| = \frac{8}{27} e^{5/6} \Lambda_{\text{QCD}}^{\overline{\text{MS}}} \simeq 0.7 \Lambda_{\text{QCD}}^{\overline{\text{MS}}}. \quad (59)$$

Thus, the perturbative expansion *per se* anticipates the onset of the nonperturbative regime (the impossibility of locating the would-be quark pole to accuracy better than Λ_{QCD}). Certainly, the concrete numerical value of the uncertainty in m^{pole} obtained through renormalons is not trustworthy. The renormalons do not represent the dominant component of the infrared dynamics. However, they are a clear indicator of the presence of the power-suppressed nonperturbative effects, or infrared instability of m^{pole} ; the very fact that there is a correction $\mathcal{O}(\Lambda_{\text{QCD}}/m_Q)$ is beyond any doubt.

It is worth noting that the pole mass was the first example where a quantity which is perturbatively infrared-stable was shown not to be stable nonperturbatively at the level Λ_{QCD}^1 . The observation of Refs. [14, 15] gave impetus to dedicated analyses of other perturbatively infrared-stable observables in numerous hard processes without OPE, in particular, in jet physics. Such nonperturbative infrared contributions, linear in Λ_{QCD}/Q were indeed found shortly after in thrust and many other jet characteristics (for a review and a representative list of references see e.g. [17]).

Since it is impossible to relate $m_Q(\mu)$ and m_Q^{pole} to the necessary accuracy, it is clear that either m_Q^{pole} or $m_Q(\mu)$ must be irrelevant for the $1/m_Q$ expansion, for example, for calculating the decay widths of heavy flavors which depend on high powers of the quark masses. Which mass must be used then? This question was formulated and answered in the early 1994 in Ref. [14]. In agreement with the qualitative discussion given above, the answer is: m_Q^{pole} is irrelevant and must be replaced everywhere by $m_Q(\mu)$. For the inclusive decay widths, this can be understood as the perturbative counterpart of the QCD theorem [18] stating that the mass gap $\sim \mathcal{O}(\Lambda_{\text{QCD}}^1)$ differentiating the hadron mass from the quark mass, does not affect the width – the pole mass dressed by soft interactions is the perturbative analogue of the hadron mass. The $1/m_Q$ infrared renormalon uncertainty in the pole mass disappears from the perturbative corrections when the width is expressed in terms of the short-distance heavy quark mass; it is present when one attempts to compute the width through the pole mass.

Another transparent way to illustrate irrelevance of the pole mass is to vary the number of space dimensions, *viz.* to descend to the (2+1) theory. In $D = 3$ the (pole) mass logarithmically diverges in the infrared, cf. Eq. (54). The OPE, at the same time guarantees that the total widths are infrared safe in any dimension, so this infrared part cannot affect the width. The infrared divergence of the mass in $D = 3$ is simply reflection of the Coulomb potential $\ln R$ growing at large R . It is evident that total decay probabilities of a heavy quark must know nothing about the behavior of the interaction at infinite distance.

In the OPE, the infrared part of the pole mass is not related to any local operator $\bar{Q}O_i Q$, and does not enter any observable calculable in the short-distance expansion. Since this infrared piece does not enter observables, it cannot, in turn, be determined experimentally. The numerical instability of various attempts to pinpoint the value of m_Q^{pole} is a result of using the perturbative expansion for the effects originating from the nonperturbative domain. The above facts were later illustrated in Ref. [19] in a concrete model for the higher-order corrections to the semileptonic widths obtained

via the so-called bubble resummation of the one-loop perturbative diagrams.

It is important to note that the irrelevance of the pole mass goes beyond the problems with the infrared renormalon contributions illustrated above. Even if there existed some way to define reasonably the sum of the pure perturbation series for m_Q^{pole} , or asymptotic states with single-quark quantum numbers with finite energy existed in a strong-interaction theory like QCD, this mass still would have been inadequate for constructing the effective field theory, to the extent the difference with $m_Q(\mu)$ cannot be neglected numerically.

Summarizing, the pole mass *per se* does not appear in OPE for infrared-stable quantities. Such expansions operate with the short-distance (running) mass. Any attempt to express the OPE-based results in terms of the pole mass creates a problem making the Wilson coefficients ill-defined theoretically and poorly convergent numerically.

A properly constructed perturbative treatment suitable for the $1/m_Q$ expansion incorporates only gluons with $|\vec{k}| \gtrsim \mu$ which are not ‘resolved’ and are included into the heavy quark field wavefunction corresponding to the heavy quark field $Q_{(\mu)}(x)$ normalized at the scale μ . The normalization point μ can be changed: descending from μ down to $\mu_1 < \mu$ one has to integrate out newly-unresolved gluons with $\mu_1 < |\vec{k}| < \mu$. For example, the Coulomb field associated with such gluons increases the mass of the quark by the amount

$$\delta m_Q = \int_{\mu_1 < |\vec{k}| < \mu} \frac{d^3 \vec{k}}{4\pi^2} \frac{4}{3} \frac{\alpha_s(\vec{k}^2)}{\vec{k}^2}. \quad (60)$$

The pole mass clearly appears when $\mu_1 \rightarrow 0$. From the OPE point of view, it is an attempt to construct an effective theory with the normalization scale $\mu = 0$ formulated, nevertheless, still in terms of quarks and gluons. Speaking theoretically, one can imagine a limit of small μ which would correspond to integrating out all modes down to $\mu = 0$ in evaluation of the effective Lagrangian. It would be nothing but constructing the S -matrix of the theory from which one can directly read off all conceivable amplitudes. Clearly, it could have been formulated in terms of physical mesons and baryons but not of quarks and gluons.

In the Wilson OPE one uses $m_Q(\mu)$ with $\Lambda_{\text{QCD}} \ll \mu \ll m_b$. We illustrate later that just such a mass can be accurately measured in experiment. It *is* μ -dependent:

$$\frac{dm_Q(\mu)}{d\mu} = -\frac{16}{9} \frac{\alpha_s(\mu)}{\pi} - \frac{4}{3} \frac{\alpha_s(\mu)}{\pi} \frac{\mu}{m_Q} + \mathcal{O}\left(\alpha_s^2, \alpha_s \frac{\mu^2}{m_Q^2}\right); \quad (61)$$

the higher order perturbative corrections were computed recently [10]. There are different schemes for defining $m_Q(\mu)$ (similar to renormalization schemes for α_s), and the coefficients above are generally different there. As long as a concrete scheme is adopted, there is no ambiguity in the numerical value of $m_Q(\mu)$. Instead of the HQET parameter $\bar{\Lambda}$ in QCD one has $\bar{\Lambda}(\mu) = \lim_{m_Q \rightarrow \infty} M_{HQ} - m_Q(\mu)$. The value of $\bar{\Lambda}(\mu)$ is of the hadronic mass scale if μ does not scale with m_Q .

There exists a popular choice of a short-distance mass, the so-called $\overline{\text{MS}}$ mass $\bar{m}(\mu)$. The $\overline{\text{MS}}$ mass is not a parameter in the effective Lagrangian; rather it is a

certain *ad hoc* combination of the parameters which is particularly convenient in the perturbative calculations using dimensional regularization. Its relation to the perturbative pole mass since recently is known already to three loops [20]:

$$m_Q^{\text{pole}} = \bar{m}_Q(\bar{m}_Q) \left\{ 1 + \frac{4}{3} \frac{\alpha_s(\bar{m}_Q)}{\pi} + (1.56b - 3.73) \left(\frac{\alpha_s}{\pi} \right)^2 + \dots \right\} \quad (62)$$

At $\mu \gtrsim m_Q$ the $\overline{\text{MS}}$ mass coincides, roughly speaking, with the running Lagrangian mass seen at the scale $\sim \mu$. However, it becomes rather meaningless at $\mu \ll m_Q$:

$$\bar{m}_Q(\mu) \simeq \bar{m}_Q(\bar{m}_Q) \left\{ 1 + \frac{2\alpha_s}{\pi} \ln \frac{m_Q}{\mu} \right\}. \quad (63)$$

It logarithmically diverges when $\mu/m_Q \rightarrow 0$. For this reason $\bar{m}(\mu)$ is not appropriate in the heavy quark theory where the possibility of evolving down to a low normalization point, $\mu \ll m_Q$, is crucial. Otherwise, for example, $M_{H_Q} - \bar{m}_Q \propto m_Q$ and does not stay constant in the heavy quark limit.

The reason for this IR divergence is that the $\overline{\text{MS}}$ scheme technically attempts to determine the (perturbative) running of all quantities by their divergences calculated when the space-time dimension approaches $D = 4$. The divergence can emerge only at $k \rightarrow 0$ or $k \rightarrow \infty$. For the mass, the IR divergences are absent, and the dimensional regularization is sensitive to the UV divergence at $k \gg m_Q$. Studying only $1/(D-4)$ singularities, it is unable to capture the change of the regime at $\mu \lesssim m_Q$ and assumes the same running in this domain. The actual running below m_Q is slower.

The properly defined short-distance masses always exhibit an explicit linear μ -dependence similar to Eq. (61) at $\mu \ll m_Q$. The perturbative pole mass, order by order, would correspond to the limit $\mu \rightarrow 0$. However such a limit does not exist.

Since m_Q^{pole} does not exist as a well-defined mass parameter, a different, short-distance mass must be used. The normalization point μ can be arbitrary as long as $\mu \gg \Lambda_{\text{QCD}}$. It does not mean, however, that all masses are equally practical, since the perturbative series are necessarily truncated after a few first terms. Using an inappropriate scale makes numerical approximations bad. In particular, relying on $\bar{m}_Q(m_Q)$ in treating the low-scale observables can be awkward. The pedagogical example illustrating this point can be found in Refs. [21, 2].

Needless to say, it is the high-scale masses that appear directly in the processes at high energies. In particular, the inclusive width $Z \rightarrow b\bar{b}$ is sensitive to $m_b(M_Z)$; using $\overline{\text{MS}}$ mass normalized at $\mu \sim M_Z$ is appropriate here. On the contrary, the inclusive semileptonic decays $b \rightarrow c \ell \nu$ are rather low-energy in this respect [22]; this is true even for $b \rightarrow u$.

The construction of the running low-scale heavy quark mass suitable for the OPE to any order in perturbation theory can be done in a straightforward way. However, the Wilsonian approach implies introducing a cutoff on the momenta of the gluon fields. Since gluon carries color, its momentum is not a gauge-invariant quantity, and such a mass typically looks not gauge-invariant. More accurately, obtaining the same $m_Q(\mu)$ requires somewhat different cutoff rules in different gauges.

Even though this is not a real problem for the theory, it is often viewed as a disadvantage. To get rid of this spurious problem, a manifestly gauge-invariant definition of the running mass was suggested in [22, 23] which is formulated only in terms of observables. The idea is to explicitly subtract from the pole mass the infrared pieces it contains. These are uniquely determined by the so-called small velocity (SV) sum rules in the heavy quark limit discussed in Sect. 4. They are moments of the structure functions of the infinitely heavy quark in the process where its velocity changes by a small amount. In the presence of hard gluons these moments diverge in the ultraviolet, and must be cut off at some energy μ . This cutoff enters then as the normalization point for the heavy quark mass.

Such defined mass is convenient for the OPE in the $1/m_Q$ expansion since more or less directly enters many relevant processes, *e.g.* heavy flavor transitions or the threshold heavy flavor production. Its relation to the $\overline{\text{MS}}$ mass is known with enough accuracy [10]. To avoid ambiguities we always use this definition unless other convention is indicated explicitly.

It is important to note the following fact. In the relativistic theory for a particle with mass m one always has $p^2 = m^2$, that is, $E = \sqrt{m^2 + \vec{p}^2}$. In the nonrelativistic expansion, therefore,

$$E = m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3} + \dots; \quad (64)$$

all coefficient are fixed in terms of powers of m by Lorentz invariance. In applications to heavy quarks it is advantageous to use such a Wilsonian cutoff which preserves usual QM properties for the price of apparently violating Lorentz invariance. This is justified since the problem from the very beginning has physically preferred frame – the one in which the heavy quark is at rest. (More detailed discussion can be found in [23].) Then the mass parameters in Eq. (64) generally become different:

$$\mathcal{H}_Q = m_0 \varphi_Q^+ \varphi_Q - \varphi_Q^+ A_0 \varphi_Q + \frac{1}{2m_2} \left((i\vec{D})^2 + c_G \vec{\sigma} \vec{B} \right) - \dots \quad (65)$$

even for a “quasifree” quark in the effective theory. This phenomenon is well known in the solid state physics. Of course, the difference between m_i appears only due to perturbative corrections:

$$m_i(\mu) - m_k(\mu) = \mathcal{O}(\alpha_s \mu).$$

Moreover, the differences can be calculated perturbatively and are completely free from any infrared effects existing below the scale μ . For example, in our scheme $m_0(\mu) \simeq m_2(\mu) + \frac{4}{9} \frac{\alpha_s(\mu)}{\pi}$. We use the mass $m_2(\mu)$, the mass that enters the kinetic energy operator $\vec{p}^2/2m$; it is the most relevant mass for heavy quark transitions.

This calculable difference between different “masses” for the same quark must be properly accounted for in the OPE analysis. Probably, the most obvious place where it plays a role is the $\bar{Q}Q$ threshold physics. While in the short-distance expansion the free quark threshold starts at $2m_0(\mu)$, the propagation of heavy quarks or the bound state dynamics are actually determined by $m_2(\mu)$. The shift in the position

of the threshold which serves as a reference point for energy in the nonrelativistic system must be properly taken into account.

Concluding this section, let us make a side remark concerning the t quark mass [21]. The peculiarity of the t quark is that it has a significant width $\Gamma_t \sim 1\text{ GeV}$ due to its weak decay. The perturbative position of the pole in the propagator is, thus, shifted into the complex plane by $-\frac{i}{2}\Gamma_t$. The finite decay width of the t quark introduces a *physical* infrared cutoff for the infrared QCD effects [24]. In particular, the observable decay characteristics do not have ambiguity associated with the uncertainty in the pole mass discussed above. The uncertainty cancels in any physical quantity that can be measured. That is not the case, however, in the position of the pole of the t -quark propagator in the complex plane (more exactly, its real part). The quark Green function is not observable there, and one would encounter the very same infrared problem and the same infrared renormalon. The latter does not depend on the absolute value of the quark mass (and whether it is real or have an imaginary part). Thus, in the case of top, one would observe an intrinsic – but artificial – infrared renormalon uncertainty of several hundred MeV in attempts to relate the peak in the physical decay distributions to the position of the propagator singularity in the complex plane.

3.1.2 The numerical values of m_c and m_b

The mass of the c quark at the scale $\sim m_c \sim 1\text{ GeV}$ can be obtained from the charmonium sum rules [25], $\bar{m}_c(m_c) \simeq 1.25\text{ GeV}$. The result to some extent is affected by the value of the gluon condensate. To be safe, we conservatively ascribe a rather large uncertainty,

$$m_c(m_c) = 1.25 \pm 0.1\text{ GeV}.$$

There are reasons to believe that the precision charmonium sum rules actually determine the charmed quark mass to a better accuracy.

An accurate measurement of m_b is possible in the $\bar{b}b$ production in e^+e^- annihilation. Since we want to know m_b with comparable or better absolute precision, both the data and calculations, at first glance, must have increased accuracy. The data are available, however, only below and near the threshold. Certain integrals (moments) of the cross section over this domain are particularly sensitive to the low-scale mass $m_b(\mu)$ with μ in the interval 1 to 2 GeV.

Using dispersion relations

$$\Pi_b(q^2) = \Pi_b(0) + \frac{q^2}{2\pi^2} \int \frac{ds R_b(s)}{s(s-q^2)} \quad (66)$$

one evaluates the polarization operator $\Pi_b(q^2)$ (and its derivatives) induced by the vector currents $\bar{b}\gamma_\mu b$, in the complex q^2 plane at an adjustable distance Δ from the threshold. Such quantities are proportional to weighted integrals over the experimental cross section; the integrals are saturated in the interval $\sim \Delta$ near threshold, and are very sensitive to the mass $m_b(\Delta)$.

A dedicated analysis of this type was first carried out by Voloshin [26] who considered a set of relatively high derivatives of Π_b at $q^2=0$. On the phenomenological side they are expressed through moments of $R_b(s)$,

$$\frac{2\pi^2}{n!} \Pi_b^{(n)}(0) = I_n = \int \frac{ds R_b(s)}{s^{n+1}} \simeq M_{\Upsilon(1S)}^{-2(n+1)} \int ds R_b(s) \exp \left\{ -(n+1) \left(s - 4M_{\Upsilon(1S)}^2 \right) \right\}, \quad (67)$$

while the theoretical expressions for the very same moments are given in terms of the b quark mass and α_s . The relevant momentum scale here is $\mu \sim m_b/\sqrt{n}$. Considering small- n moments I_n one would determine m_b at the scale of the order m_b . The small- n moments are contaminated by the contribution of R_b above the open beauty threshold where experimental data are poor. Increasing n shrinks the interval of saturation and, thus, lowers the effective scale. On the other hand, we cannot go to too high values of n where infrared effects (given, first, by the gluon condensate) explode. There is still enough room to keep the gluon condensate small and, simultaneously, suppress the domain above the open beauty threshold. In the fiducial window, μ must be large enough to ensure control over the QCD corrections. The latter requires a nontrivial summation of enhanced Coulomb terms unavoidable in the nonrelativistic situation. As known from textbooks, the part of the perturbative corrections to the polarization operator, associated with the potential interaction, is governed by the parameter $\alpha/|\vec{v}|$ rather than by α *per se*.

Let us briefly illustrate why the moments I_n determined in this way pinpoint the running mass m_b . Naively, the cut in $\Pi_b(q^2)$ starts in the perturbative calculations at $2m_b^{\text{pole}}$, which seems to determine the strongest dependence on the b quark mass. However, the $\bar{b}b$ production is also affected by the potential interaction of b and \bar{b} ; the latter even generates a number of bound states below the $\bar{b}b$ threshold. The gluon exchanges increase m_b^{pole} and thus tend to suppress I_n (we keep the short distance mass fixed). However, the Coulomb effects enhance the moments both owing to the emerging bound states and due to attraction above the threshold.

The largest infrared contribution to the pole mass is linear in the gluon momentum, Eq. (54). Its cancellation can be understood on the example of QED with the Abelian gauge interaction. The expression for the mass shift is simply self-interaction $\frac{1}{2}V_{\text{IR}}(0)$ where V_{IR} is the heavy quark potential mediated by the gauge interactions with momenta below certain $\mu \ll m_b$ [13, 14]. Yet the mass of the $\bar{b}b$ system includes also the same Coulomb interaction between quark and antiquark. Since for the colorless $\bar{b}b$ state the sum of color “charges” is zero, these effects cancel each other for the quanta with wavelength less than the interquark spacing r . Therefore, for the Fourier transform of the potential $V(\vec{q})$ in terms of which

$$V(0) = \int \frac{d^3\vec{q}}{(2\pi)^3} V(\vec{q}), \quad (68)$$

only the components with $|\vec{q}| \gtrsim 1/r$ contribute. The softer exchanges are suppressed by powers of the multipole factor $\vec{q}^2 r^2$.

This, of course, automatically emerges in all calculations. Let us single out the effect of gluon exchanges with $|\vec{q}| < \mu$:

$$V_{\text{IR}}(r) = - \int_{|\vec{q}| < \mu} \frac{d^3 \vec{q}}{(2\pi)^3} V(\vec{q}) e^{-i\vec{q}\vec{r}} = -V_0 + \frac{1}{6} r^2 \mu^2 V_2 - \dots, \quad (69)$$

$$V_0 = \int_{|\vec{q}| < \mu} \frac{d^3 \vec{q}}{(2\pi)^3} V(\vec{q}), \quad V_2 = \int_{|\vec{q}| < \mu} \frac{d^3 \vec{q}}{(2\pi)^3} \frac{\vec{q}^2}{\mu^2} V(\vec{q}), \quad \dots$$

(the minus sign reflects the fact that the second particle is an antiquark). If quarks reside at distances much smaller than $1/\mu$, the soft potential is just a constant. Its sole role is only to shift the energy of all $\bar{b}b$ states by a constant amount $-V_0$. It does not affect wavefunctions and, therefore, does not modify the coupling of the virtual photon in e^+e^- annihilation to these states.

However, a constant potential A_0 cannot change the energy of the neutral system, whether the field is classical or quantum. It means that just the opposite shift is present in the sum of masses of b and \bar{b} renormalized by the same gauge interaction, which is self-manifest in Eq. (54). The above arguments show that the cancellation of the infrared effects in the quark mass is a nontrivial consequence of the gauge nature of QCD interactions, the fact explicit in the OPE.

The above reasoning, while illustrating the cancellation of the infrared effects present in the pole mass, is too simplified in many aspects. A more dedicated discussion can be found in Ref. [2].

Over the last few years the perturbative description of the moments I_n has been significantly improved and now includes the next-to-next-to-leading (NNLO) terms. The corresponding analysis of the data was performed by a few groups [27], differing in technical details; the values of $m_b(1 \text{ GeV})$ were found around 4.57 GeV with the estimated uncertainty 40 to 60 MeV. In what follows we accept the value of the running b quark mass

$$m_b(1 \text{ GeV}) = 4.57 \pm 0.05 \text{ GeV} \quad (70)$$

3.2 μ_π^2 and μ_G^2

The heavy quark masses m_c, m_b , being the key parameters in the HQE, are to a large extent ‘external’ to the properties of the effective low energy theory itself. There are two nonrelativistic heavy quark operators in the Hamiltonian; their expectation values Eqs.(22) in the heavy meson B play a key role in many applications. In contrast to m_Q , they are determined by the QCD dynamics itself. We cannot yet calculate theoretically their values from first principles of QCD since it would require more or less exact solution of QCD in the strong coupling regime. Instead, we can try to measure them extracting from known properties of hadrons.

The value of μ_G^2 is known: since

$$\frac{1}{2m_Q} \bar{Q} \frac{i}{2} \sigma_{\mu\nu} G^{\mu\nu} Q$$

describes the interaction of the heavy quark spin with the light cloud and causes the hyperfine splitting between B and B^* ,

$$\mu_G^2 \simeq \frac{3}{4} 2m_b(M_{B^*} - M_B) \simeq \frac{3}{4} (M_{B^*}^2 - M_B^2) \approx 0.36 \text{ GeV}^2. \quad (71)$$

In actual QCD μ_G^2 logarithmically depends on the normalization point; usual one-loop diagrams yield

$$\mu_G^2(\mu') \simeq \left(\frac{\alpha_s(\mu')}{\alpha_s(\mu)} \right)^{\frac{3}{b}} \mu_G^2(\mu). \quad (72)$$

In the mass relation given above the operator is normalized at the scale $\mu \simeq m_b$:

$$\frac{1}{m_Q} \mathcal{H}_1^{\text{spin}} = -C_G(\mu) \frac{(\bar{Q} \frac{i}{2} \sigma G Q)_\mu}{2m_Q}, \quad C_G \simeq \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^{\frac{3}{b}}. \quad (73)$$

Evolving perturbatively to the normalization scale $\mu \sim 1 \text{ GeV}$ slightly enhances the value of $\mu_G^2(\mu)$, but this effect is numerically insignificant, and we usually neglect it.

The kinetic expectation value μ_π^2 has not been directly measured yet. It enters various distributions in semileptonic decays or processes of the type $b \rightarrow s + \gamma$. First attempts to extract it from the semileptonic distributions were reported [28, 29], however the analysis is vulnerable at some points, and so far the outcome is inconclusive.

A model-independent lower bound was established [30, 31, 9, 4]

$$\mu_\pi^2 > \mu_G^2 \approx 0.4 \text{ GeV}^2 \quad (74)$$

which constrained possible values of μ_π^2 . One of its possible derivations will be given below in Sect. 4 (see also Sect. 5.1). Here we instead mention its physical interpretation.

Unlike quantum-mechanical pure potential problems, in QCD the heavy quark kinetic operator is expressed in terms of the *covariant* derivatives $\pi_j = iD_j = i\partial_j + A_j$ which include the vector potential A_j . The latter lead to non-commutativity of different spatial components of the momentum operator in the presence of the chromomagnetic field \vec{B} ,

$$[\pi_j, \pi_k] = iG_{jk} = -i\epsilon_{jkl} B_l \simeq -\frac{2}{3} i\epsilon_{jkl} j_l \cdot 0.4 \text{ GeV}^2. \quad (75)$$

This non-commutativity immediately leads to the lower bound on the expectation value of $\vec{\pi}^2$ [30], in full analogy with the uncertainty principle in quantum mechanics. Formally, it follows from the positivity of the Pauli Hamiltonian [31]:

$$\frac{1}{2m} (\vec{\sigma} i\vec{D})^2 = \frac{1}{2m} \left((i\vec{D})^2 - \frac{i}{2} \sigma G \right) > 0.$$

More physically, it means the Landau precession of a colored, i.e. “charged” particle in the (chromo)magnetic field, Fig. 3. Hence, one has $\langle p^2 \rangle \geq |\vec{B}|$. Literally speaking,

in the B^* meson the quantum-mechanical expectation value of the chromomagnetic field is suppressed, $\langle B_z \rangle = -\mu_G^2/3$. It completely vanishes in the B meson. However, the essentially non-classical nature of the ‘commutator’ \vec{B} proportional to the spin operator of the light cloud (e.g. $\langle \vec{B}^2 \rangle \geq 3\langle \vec{B} \rangle^2$), in turn, enhances the bound which then takes the same form as in the external classical field.

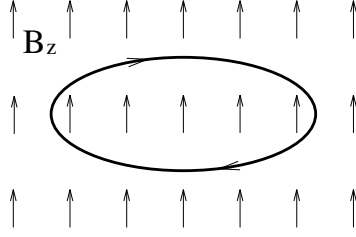


Figure 3: Landau precession of a free charged particle in the magnetic field. The average of the momentum square is bounded from below even in the absence of binding potential. This illustrates the physical meaning of the inequality between μ_π^2 and μ_G^2 .

It is worth emphasizing that the inequality between the kinetic and chromomagnetic expectation values takes place for μ_π^2 normalized at any point μ , provided μ_G^2 is normalized at the same point. For large μ it becomes uninformative; so, it is in our best interests to use it at $\mu = \text{several units} \times \Lambda_{\text{QCD}} \lesssim 1 \text{ GeV}$.

The B meson average of the kinetic operator was estimated using the technique of the QCD sum rule in [32, 33] and in [34], with quite different results $(0.5 \pm 0.15) \text{ GeV}^2$ and $0.1 \pm 0.05 \text{ GeV}^2$, respectively. A dedicated discussion of the subtleties inherent to those analyses can be found in Ref. [21], Sect. 6. The discrepancy roots to a different treatment of the contributions of the excited states which are a background in applications of the QCD sum rules, and the actual result seemingly lies somewhere in between the two extremes, probably closer to the upper value. It was later argued in Ref. [35] based on the analysis of the QCD sum rule approach to the three-point correlators in the harmonic oscillator, that using the same continuum threshold for the two- and three-point correlators (which is routinely assumed in QCD sum rules) leads to a systematic bias, and that it is advantageous to use a lower continuum threshold for the three-point functions. Applying this prescription to modify the analysis of Ball *et al.*, the authors got a value close to 0.4 GeV^2 .

It is important to note, however that the normalization point dependence of the kinetic expectation value was not treated consistently. In fact, the above papers attempted to determine the HQET parameter $-\lambda_1$ rather than the properly defined QCD expectation value μ_π^2 dependent on the normalization point. In simple QM models with fixed numbers of constituents in a hadron, one would have them equal, $-\lambda_1 = \mu_\pi^2$. However, in actual QCD $-\lambda_1$ can be thought of as the kinetic expectation value $\mu_\pi^2(\mu)$ (μ is the normalization point of the operator) from which “all perturbative” (μ -dependent) pieces are subtracted. In this respect $-\lambda_1$ is a close relative of the pole mass, and likewise cannot be defined consistently in reality. Techni-

cally, it corresponds to extrapolating the normalization point in $\mu_\pi^2(\mu)$ down to zero momentum, which leads to the usual conceptual problems. Related questions are discussed in more detail later in Sect. 4.2 and in Ref. [2], Sect. 3. In practical terms, since the above mentioned QCD sum rules analyses were performed at the level of one-loop $\mathcal{O}(\alpha_s)$ perturbative corrections, it is more or less sufficient to use identification $-\lambda_1 \rightarrow -\lambda_1 + \frac{4\alpha_s}{3\pi}\mu^2 \simeq \mu_\pi^2(\mu)$. This suggests adding about 0.15 GeV^2 to the quoted values of $-\lambda_1$ to arrive at the QCD kinetic expectation value normalized at the canonical scale around 0.7 to 1 GeV.

To summarize, the value of the kinetic expectation value μ_π^2 in B mesons determined from the QCD sum rules is about

$$\mu_\pi^2 \simeq 0.5 \pm 0.15 \text{ GeV}^2 , \quad (76)$$

however the literally quoted intrinsic uncertainty of the method may be underestimated here. It is noteworthy that this estimate seems to be in a good agreement with the bound Eq. (74). More details regarding various subtleties in the kinetic operator can be found in Ref. [2].

3.3 Heavy quark potential

We discuss here briefly some aspects related to the heavy quark potential in QCD. Strictly speaking, it refers to a somewhat different situation where two heavy quarks (actually, a quark and an antiquark) are present simultaneously. Heavy quark potential attracted recently attention in connection to pair production of $t\bar{t}$ and $b\bar{b}$ near the threshold, and in particular in view of its apparent connection to the problem of the heavy quark mass illustrated in Sect. 3.1.

A closer look at the notion of the heavy quark potential reveals certain subtleties. As a matter of fact, it is not completely clear what exactly must be called the potential between heavy quarks in actual QCD.

The original notion of the potential refers to the interaction of infinitely heavy (static), or completely nonrelativistic heavy particles, which is *instantaneous*. The most familiar example is the electromagnetic interaction of heavy charges. The Hamiltonian of such a potential system is given by

$$\mathcal{H} = \sum_i \frac{(i\vec{\partial})^2}{2m_i} + V(\vec{r}_i - \vec{r}_j) , \quad (77)$$

where m_i are masses of particles and the potential V is a function of their instant coordinates. Taking the limit $m_i \rightarrow \infty$ (at fixed \vec{r}_i , which corresponds to semi-classically high excitation numbers of the quantum system in Eq. (77)) eliminates quantum uncertainties in the coordinates and allows to measure the potential directly as the position-dependent energy of the infinitely slowly moving collection of particles. This is well known for QED where the potential of the charges q_1 and q_2 (in units of electron charge) is given by

$$V^{\text{QED}}(R) = \alpha_{\text{em}}(0) \frac{q_1 q_2}{R} . \quad (78)$$

This expression is exact in the absence of light charged particles; the known quantum corrections appear only if other matter fields are not much heavier than the scale $1/R$.

The definition of the similar quantity in a non-Abelian theory like QCD is more tricky. First, one has to limit consideration to the systems in the colorless state. The heavy quark potential is defined only between Q and \bar{Q} in the color singlet case; even there the requirement of gauge invariance is not trivial. The color of the individual heavy quark remains fully quantum in nature and changes through the interaction with gluons. In contrast to usual coordinates, the limit $m_Q \rightarrow \infty$ does not make color variables describing the state of the heavy quark semiclassical.

To avoid this problem the heavy quark potential is defined *via* the vacuum expectation value of the long Wilson loops:

$$V(R) = - \lim_{T \rightarrow \infty} \frac{1}{T} \ln \langle \text{Tr } \mathcal{P} \exp \left(i \oint_{C(R,T)} A_\mu dx_\mu \right) \rangle, \quad (79)$$

where the rectangular contour $C(R, T)$ spans distance R and T in the space and time directions, respectively. (It is usually assumed to be in the Euclidean space as in Eq. (79).) This definition is intuitively clear, since Wilson lines describe propagation of the infinitely heavy quarks. Moreover, such Wilson loops are readily computed in the $U(1)$ gauge theory (free QED) and, of course, reproduce Eq. (78) (with $q_1 = -q_2$). In QCD the perturbative expansion of $V(R)$ has been computed through order α_s^3 [36]. Usually the potential in the momentum representation is considered:

$$V(\vec{q}) = \int d^3 \vec{R} V(\vec{R}) e^{-i\vec{q}\vec{R}} = -\frac{4}{3} \frac{4\pi\alpha_s}{\vec{q}^2} \left\{ 1 + \left(\frac{31}{3} - \frac{10}{9}n_f \right) \frac{\alpha_s}{4\pi} + c_3 \left(\frac{\alpha_s}{4\pi} \right)^2 + \dots \right\}. \quad (80)$$

There are important peculiarities in thus defined interaction in the $Q\bar{Q}$ system; they were first analyzed by Appelquist *et al.* already in the late 70's [37]. Due to gluon self-interaction, the potential in higher orders contains diagrams of the type shown in Figs. 4. Here the dashed line denotes the Coulomb quanta (they mediate instantaneous interaction in the physical Coulomb gauge), while the wavy lines are used for actual transverse gluons. These diagrams signal the real propagation in time of transverse gluons. This means that at this level the problem ceases to be a two-body one, and includes more full-fledged dynamical degrees of freedom.

Moreover, while the first diagram Fig. 4a with the rung gluon appearing in order α_s^3 safely converges at the gluon momenta $\sim 1/R$, including additional Coulomb exchanges, as in Fig. 4b leads to infrared divergence. With one exchange it is logarithmic; the formal degree of the infrared divergence increases with adding extra Coulomb quanta between the emission and absorption of the transverse gluon. The physics behind this infrared behavior was discussed in Ref. [37]: emission of the soft transverse gluon changes the overall color of the $Q\bar{Q}$ pair and, therefore modifies the interaction energy between them. The energy shift associated with the exchange of the transverse gluon depends nonanalytically on the energy denominator (in the language of non-covariant time-ordered perturbation theory), since the gluon can

be arbitrarily soft. Formally expanding the exact result in α_s includes expanding in this change of the Coulomb energy proportional to α_s/R , therefore one obtains increasing infrared singularities. These arguments suggested that resummation of the Coulomb exchanges in Fig. 4b would render these diagrams finite, however the effective infrared cutoff is of the order of α_s/R and, thus the potential is not infrared finite *perturbatively* containing terms $\sim 1/R \cdot \ln \alpha_s$ starting order $\mathcal{O}(\alpha_s^4)$.

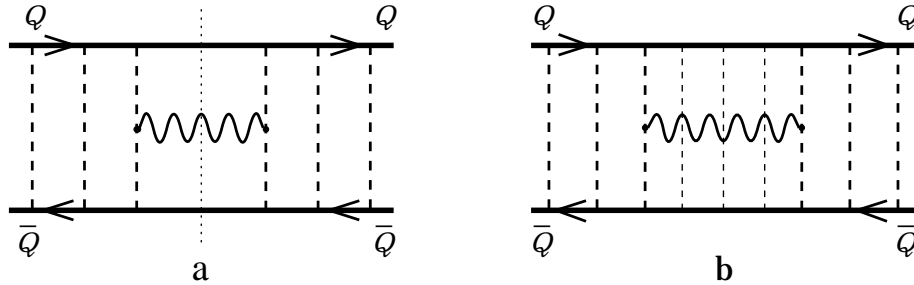


Figure 4: Diagrams for heavy quark potential in QCD. Dashed lines are instantaneous Coulomb exchanges, transverse gluons propagating in time are shown by wavy lines.

a) The convergent diagrams with the transverse gluon as a rung.

b) Adding more Coulomb exchanges inside the ladder with the transverse gluon leads to infrared divergence in perturbation theory.

This infrared singularity emerges since the external momentum scale $1/R$ does not fix all (spacelike) momenta in the problem. In the static situation the low-momentum tail in the propagation of very low-momentum (ultrasoft) gluons is cut off only by the interaction energy of the quarks, which appears itself only in the perturbative expansion; therefore it turns out to be suppressed only by powers of α_s . As a result, beyond order α_s^3 the heavy quark potential, whatever it means, is not a short-distance quantity even at arbitrary small space separation between the heavy quarks.⁵ Of course, this becomes possible only since time interval used to define $V(R)$ is taken infinitely large. Should one use instead finite- T Wilson loops, the time interval would place the final infrared cutoff in all gluon exchanges.

This purely perturbative analysis shows that there is no direct analogue of the potential between heavy quarks in QCD. The $Q\bar{Q}$ system incorporating all gluon interaction is not a two-body system but includes actual propagation of gluons with energy small compared to $1/R$. The interaction then cannot be universally described by an instantaneous potential and is intrinsically non-local in time. An attempt to integrate out in one way or another the “extra” degree of freedom does not yield in general a meaningful analogue of the potential: the result would not be universal but rather process-dependent. Thus, the prescription (79) based on Wilson loops in QCD does not yield the heavy quark potential in its conventional understanding.

⁵This situation is not peculiar to non-Abelian gauge theories. Similar long-range propagation of light charged particles occurs also in QED if their masses are well below $1/R$.

In practice, this may not pose serious problems in the perturbative computations. Through order α_s^3 all diagrams including Fig. 4a are convergent and saturated by the gluon momenta of order $1/R$. The above nonlocalities in time are then revealed only at time intervals smaller than R . In many applications with nonrelativistic heavy quarks the scale $1/R$ is still much smaller than the characteristic dynamic energies. In the spirit of the Wilsonian OPE such exchanges then can be integrated out to yield a local in time effective potential. Yet it must be remembered that this can be done only up to a certain order in α_s , depending on the concrete problem.

As discussed above, the heavy quark potential V (from now on we assume that it is defined via Wilson loops, Eq. (79)) is not an infrared finite object already in the perturbative expansion. The situation can be better for the *integral* of $V(\vec{q})$ over spacelike momentum:

$$V(0) = \int \frac{d^3\vec{q}}{(2\pi)^3} V(\vec{q}) . \quad (81)$$

In order to get a finite result we need to assume an ultraviolet cutoff at a certain scale μ , for example, adding the step-function $\theta(\mu^2 - \vec{q}^2)$ to the integrand. It was mentioned in Sect. 3.1 that in electrodynamics such an integral of the potential equals minus double of the contribution to the mass of the static source coming from the same range of gluon momenta. In QCD the perturbative diagrams for both m_Q and $V(\vec{q})$ are more complicated. Yet the same relation holds in order $\mathcal{O}(\alpha_s^2)$ as well. Most simply this is seen in the Coulomb gauge where all the effect to this order reduces to dressing the propagator of the Coulomb quanta. (An alternative discussion can be found in Ref. [38].) There are reasons to believe that such a relation may hold to all orders in perturbation theory: the infrared contribution to $V(0)$ in Eq. (81) equals minus twice the same contribution to the pole mass of a static source.

Indeed, let us imagine we were able to introduce in some way an ensemble of gauge field configurations where the modes with momenta much larger than a certain scale μ are practically absent. This field-theoretic system would not need regularization, and everything can be expressed in terms of the bare parameters, including the bare quark mass $m_Q^{(0)}$. At $R \rightarrow 0$ the Wilson loop will approach the free value N_c thus yielding $V(0)=0$. This is clear on the physical grounds: $Q\bar{Q}$ form a dipole with the infinitesimal dipole moment, and its interaction with any soft gluon field vanishes as R goes to 0. It is important at this point that our gauge ensemble explicitly includes only soft modes. Otherwise, as in full QCD, the modes with $|\vec{k}| \sim 1/R$ generate growing attractive potential at arbitrary small R .

Next we note that $V(R)$ is traditionally determined up to a constant; in perturbation theory (in four dimensions) the potential is actually defined as

$$U(R) = V(R) - V(\infty) ; \quad (82)$$

we assign U to this “standard” potential to distinguish it from $V(R)$ which has a precise meaning in a finite theory. While $U(R)$ by definition vanishes at $R \rightarrow \infty$, $V(\infty)$ does not and reflects nontrivial interaction with the gluon field. In the momentum representation $V(\vec{q})$ contains explicitly the term $V(\infty) \delta^3(\vec{q})$, which is

discarded in the usual perturbative computations. Since at $R \rightarrow \infty$ the \bar{Q} and Q lines are well separated, their interaction (at least in perturbation theory) must vanish as $1/R$, and the value of the Wilson loop is simply given by the mass renormalization of each static source:

$$V(\infty) = 2 \left(m_Q^{\text{Ren}} - m_Q^{(0)} \right) = 2\delta m_Q . \quad (83)$$

Therefore, for the “ordinary” potential we have

$$U(0) = \int \frac{d^3 \vec{q}}{(2\pi)^3} V_{\text{reg}}(\vec{q}) = -2\delta m_Q . \quad (84)$$

Here V_{reg} is the usually computed regular part of $V(\vec{q})$ not containing self-energy diagrams yielding $\delta^3(\vec{q})$.

The above relation for the infrared contributions to the mass and the potential

$$\int \frac{d^3 \vec{q}}{(2\pi)^3} V_{\text{reg}}^{\text{IR}}(\vec{q}) = -2\delta_{\text{IR}} m_Q \quad (85)$$

at first may seem strange: What happens to the perturbatively infrared-singular contributions of Fig. 4b which, at order α_s^4 , behave like $\frac{\alpha_s^4}{q^2} \ln \frac{\vec{q}^2}{\epsilon}$, with ϵ an infrared cutoff in the “rung” gluon momentum? The corresponding contributions are absent from the mass since the Coulomb exchanges are instantaneous.

The subtlety resides in the necessity to introduce an upper cutoff μ on gluon momenta in V^{IR} . This not only cuts off $V(\vec{q})$ at $\vec{q}^2 \gg \mu^2$, but also modifies $V(\vec{q})$ at $|\vec{q}| \sim \mu$ and must lead to an additional term residing around $|\vec{q}| \sim \mu$ which contains a similar term $\frac{\alpha_s^4}{\mu^2} \ln \epsilon$ with the opposite sign. They do not, of course, eliminate dependence on ϵ in the resulting $V^{\text{IR}}(\vec{q})$, but have to cancel it when integrated over all \vec{q} .

This requirement has a transparent meaning: the amplitudes of emitting a very soft gluon out of the $Q\bar{Q}$ system where initially Q and \bar{Q} are set to be at arbitrarily small separation R , must vanish unless virtual gluons with $\vec{k} \sim 1/R$ are allowed to resolve Q and \bar{Q} . Therefore, with all gluon fields softer than a certain μ , the sum of all diagrams for the emission *amplitude* (given by the part of the graphs in Figs. 4 to the left from the transverse gluon) must vanish upon integrating over \vec{q} .

Relying on relation (85), one can try to define perturbatively a certain running heavy quark mass which is free from the leading renormalon uncertainty $\sim \Lambda_{\text{QCD}}$.⁶ One defines

$$m_Q^{\text{PS}}(\mu) = m_Q^{\text{pole}} + \frac{1}{2} \int_{|\vec{q}| < \mu} \frac{d^3 \vec{q}}{(2\pi)^3} V_{\text{reg}}(\vec{q}) = m_Q^{\text{pole}} + \frac{1}{\pi} \int_0^\infty dR V(R) \left[\frac{\sin \mu R}{R} - \mu \cos \mu R \right], \quad (86)$$

where simply the literal $V(\vec{q})$ computed to a certain order in perturbation theory without an explicit cutoff is used, and the pole mass is taken to the same order in α_s .

⁶This idea was first discussed myself in 1996 and later independently advocated by M. Beneke [38].

The mass $m_Q^{\text{PS}}(\mu)$ is known as the “potential-subtracted” mass [38]. Although purely soft gluon configurations would generate ‘an infrared’ part of the potential $V(\vec{q})$ literally different from $V(\vec{q})\theta(|\vec{q}|-\mu)$ used in Eq. (86), this ansatz is hoped to include all (perturbative) infrared contributions originating from the domain sufficiently below μ , and simply subtracts some additional *ad hoc* perturbative pieces.

A technical problem would arise here starting order α_s^4 – since $V(\vec{q})$ becomes IR singular here, this routine literally fails. According to the previous discussion, the weight itself with which the infrared contributions are to be subtracted apparently gets modified by the explicit cutoff, should we implement it in practice. Therefore, they are not properly subtracted from the pole mass by prescription (86). It is conceivable that this mismatch exists already at order α_s^3 , but is not revealed explicitly due to infrared convergence of V to this order.

It should be noted that $V(0)$, although infrared finite in perturbation theory, is not a genuinely short-distance quantity (nor its μ -dependence) for which, for instance, usual OPE can be applied. Eq. (86) also shows that at arbitrary large μ it includes contribution from the potential $V(R)$ at large distances R . The nonperturbative definition of such an object therefore remains unclear. These interesting questions fall outside the scope of the present review.

4 Heavy Quark Sum Rules and Exact Inequalities in the Static Limit $m_Q \rightarrow \infty$

Discussing the basic parameters of the heavy quark theory and the properties of the actual $b \rightarrow c$ transition it is often advantageous to resort to a theoretical limit when both quark masses are asymptotically large. This usually greatly simplifies the problem since eliminates proliferating effects of various power corrections in $1/m_Q$. In this limit physics of transitions between heavy quarks without change of the heavy hadron velocity is trivial: spin degrees of freedom are decoupled, only quasi-elastic transitions (those which do not change the light cloud, and the heavy quark spin can be flipped only by the weak current) occur between the respective members of the heavy quark symmetry multiplets. The transition amplitudes of heavy quark currents between the ground and excited states vanish in the heavy quark limit.

Nontrivial physics emerges when the external weak current changes the velocity of the heavy flavor hadron. The most simple and instructive are transitions which occur in the lowest, linear in $\Delta\vec{v}$ order. This so-called small velocity (SV) limit introduced by Shifman and Voloshin in the mid 80’s as a theoretical tool for studying heavy flavor amplitudes, is also relevant for actual semileptonic decays of b to charm [39]: the typical velocity of charmed final state hadrons is rather small.

Physics of semileptonic transitions in the SV regime is described by the corresponding SV structure functions of the heavy mesons (or baryons; here we explicitly

formulate our consideration assuming the initial hadron is the ground state meson):

$$W_{\mu\nu}(q_0, \vec{q}) = \frac{1}{2\pi} \text{Im} \frac{1}{2M_{H_Q}} \langle H_Q | \int d^3x dx_0 e^{i\vec{q}\vec{x} - iq_0x_0} iT \{ J_\mu(x), J_\nu^\dagger(0) \} | H_Q \rangle \quad (87)$$

and we retain only terms through order \vec{v}^2 . $W_{\mu\nu}$ is normally expanded over invariant tensor structures. Heavy quark and SV limit lead to decrease in the number of structures for arbitrary weak currents – there is only two independent structure functions.⁷ We do not present here the corresponding formalism, but instead resort to an alternative consideration phrased in terms of individual transition amplitudes from the ground to the excited states. The analogies of the moments of the SV structure functions in this language are the so-called heavy quark sum rules.

4.1 Sum rules

The heavy quark sum rules we discuss are the following:

$$Q^2 - \frac{1}{4} = 2 \sum_m |\tau_{3/2}^{(m)}|^2 + \sum_n |\tau_{1/2}^{(n)}|^2, \quad (88)$$

$$\frac{1}{2} = 2 \sum_m |\tau_{3/2}^{(m)}|^2 - 2 \sum_n |\tau_{1/2}^{(n)}|^2, \quad (89)$$

$$\frac{\bar{\Lambda}}{2} = 2 \sum_m \epsilon_m |\tau_{3/2}^{(m)}|^2 + \sum_n \epsilon_n |\tau_{1/2}^{(n)}|^2, \quad (90)$$

$$\bar{\Sigma} = 2 \sum_m \epsilon_m |\tau_{3/2}^{(m)}|^2 - 2 \sum_n \epsilon_n |\tau_{1/2}^{(n)}|^2, \quad (91)$$

$$\frac{\mu_\pi^2}{3} = 2 \sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 + \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2, \quad (92)$$

$$\frac{\mu_G^2}{3} = 2 \sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 - 2 \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2, \quad (93)$$

$$\frac{\rho_D^3}{3} = 2 \sum_m \epsilon_m^3 |\tau_{3/2}^{(m)}|^2 + \sum_n \epsilon_n^3 |\tau_{1/2}^{(n)}|^2, \quad (94)$$

$$-\frac{\rho_{LS}^3}{3} = 2 \sum_m \epsilon_m^3 |\tau_{3/2}^{(m)}|^2 - 2 \sum_n \epsilon_n^3 |\tau_{1/2}^{(n)}|^2, \quad (95)$$

a sequence which, in principle, can be continued further. Here ϵ_k is the excitation energy of the k -th intermediate state (“ P -wave states” in the quark-model language),

$$\epsilon_k = M_{H_Q^{(k)}} - M_{P_Q},$$

⁷Strictly speaking, the second one describing the antisymmetric Lorentz structure in the infinite mass limit would require considering the nonforward scattering amplitude with the overall change of velocity, and H_Q carrying spin like the ground-state vector mesons. Alternatively, one can obtain its analogue considering the usual structure functions, for example, with vector currents to order $1/m_Q^2$ even at zero recoil, or taking more complicated weak vertices with covariant derivatives.

while the functions $\tau_{3/2}^{(m)}$ and $\tau_{1/2}^{(n)}$ describe the transition amplitudes of the ground state B meson to these intermediate states. We follow the notations of Ref. [40],

$$\frac{1}{2M_{H_Q}} \langle H_Q^{(1/2)} | A_\mu | P_Q \rangle = -\tau_{1/2} (v_1 - v_2)_\mu, \quad (96)$$

and

$$\frac{1}{2M_{H_Q}} \langle H_Q^{(3/2)} | A_\mu | P_Q \rangle = -\frac{1}{\sqrt{2}} i \tau_{3/2} \epsilon_{\mu\alpha\beta\gamma} \epsilon^{*\alpha} v_2^\beta v_1^\gamma, \quad (97)$$

where $1/2$ and $3/2$ mark the quantum numbers of the light cloud in the intermediate states, $j^\pi = 1/2^+$ and $3/2^+$, respectively, and A_μ is the axial current. Furthermore, the slope parameter ρ^2 of the Isgur-Wise function is defined as

$$\frac{1}{2M_{P_Q}} \langle P_Q(\vec{v}) | \bar{Q} \gamma_0 Q | P_Q \rangle = 1 - \rho^2 \frac{\vec{v}^2}{2} + \mathcal{O}(\vec{v}^4). \quad (98)$$

Equation (88) is known as the Bjorken sum rule [41]. Superconvergent sum rules (89) and (91) are new. Equation (90) was obtained by Voloshin [42]. The expression for μ_π^2 is the BGSUV sum rule [43]. The next one was derived in Ref. [21], as well as Eq. (95). The last two sum rules are obtained along the same lines. The sum rule for the Darwin term ρ_D^3 was first presented in [44].

Let us illustrate the simple derivation of the sum rules for μ_π^2 and μ_G^2 . Since in the heavy quark limit $\bar{Q}Q$ pairs are not produced, we will use the quantum-mechanical language with respect to Q (but not the light cloud, of course). Moreover, as discussed in Sect. 1.1 it is convenient, at the first stage to assume Q to be spinless. The Q spin effects are trivially included later. Then the lowest-lying states, the S -wave configurations corresponding to B and B^* , are spin-1/2 fermions, with two spin orientations of the light cloud. We shall denote them $|\Omega_0\rangle$; the spinor wavefunction of this state is Ψ_0 . It is obvious that

$$\langle \Omega_0 | \bar{Q} (iD_j) (iD_l) Q | \Omega_0 \rangle \equiv \frac{\mu_\pi^2}{3} \delta_{jl} \Psi_0^\dagger \Psi_0 - \frac{\mu_G^2}{6} \Psi_0^\dagger \sigma_{jl} \Psi_0 = \sum_k \langle \Omega_0 | \pi_j | n \rangle \langle k | \pi_l | \Omega_0 \rangle, \quad (99)$$

where a complete set of intermediate states is inserted. They are spin-1/2 states (of the opposite parity with respect to $|\Omega_0\rangle$), generically denoted by $\phi^{(n)}$, and spin-3/2 states $\chi^{(m)}$. We will use the Rarita-Schwinger wavefunctions for the latter, i.e. a set of three spinors χ_l obeying the constraint $\sigma_i \chi_i = 0$. The normalization of these spinors is fixed by the sum over polarizations λ

$$\sum_\lambda \chi_i(\lambda) \chi_j^\dagger(\lambda) = \delta_{ij} - \frac{1}{3} \sigma_i \sigma_j. \quad (100)$$

Defining the reduced matrix elements a_n and b_m as

$$\langle \phi^{(n)} | \pi_j | \Omega_0 \rangle \equiv a_n \phi^{(n)\dagger} \sigma_j \Psi_0, \quad \langle \chi^{(m)} | \pi_j | \Omega_0 \rangle \equiv b_m \chi_j^{(m)\dagger} \Psi_0, \quad (101)$$

where $\phi^{(n)}$ and $\chi^{(m)}$ stand for the states as well as for their wavefunctions, we get

$$\mu_G^2 = -6 \sum_n |a_n|^2 + 2 \sum_m |b_m|^2, \quad (102)$$

and

$$\mu_\pi^2 = 3 \sum_n |a_n|^2 + 2 \sum_m |b_m|^2. \quad (103)$$

These expressions can be immediately generalized to the actual case of the spin- $\frac{1}{2}$ quarks Q . The quantities a_n and b_m are to be understood as the matrix elements of $\bar{b}i\vec{D}b$ between the B meson and higher even-parity states. They are related to $\tau_{1/2}^{(n)}$ and $\tau_{3/2}^{(m)}$ as follows:

$$\tau_{1/2}^{(n)} = \frac{a_n}{\epsilon_n}, \quad \tau_{3/2}^{(m)} = \frac{1}{\sqrt{3}} \frac{b_m}{\epsilon_m}, \quad (104)$$

and, therefore,

$$\mu_\pi^2 = 3 \left(2 \sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 + \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 \right), \quad \mu_G^2 = 3 \left(2 \sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 - 2 \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 \right).$$

Note that in atomic physics the combinations analogous to $\epsilon |\tau|^2$ are called “oscillator strengths”.

Relations (104) are most easily obtained from the fact that the SV amplitudes of transitions to the corresponding states are given by the overlap $\langle n(\vec{v}) | B(\vec{v}=0) \rangle$ (in the case of spinless Q ; for spin- $\frac{1}{2}$ they are given by $\langle n(\vec{v}) | \vec{\sigma}_Q | B(\vec{v}=0) \rangle$). Then we use the general relation

$$|H_Q(\vec{v})\rangle = |H_Q(0)\rangle + \pi_0^{-1} \vec{v} \vec{\pi} |H_Q(0)\rangle + \mathcal{O}(\vec{v}^2). \quad (105)$$

which nicely elucidates the meaning of the small velocity sum rules: the operator $\pi_0^{-1}(\vec{v} \vec{\pi})$ acting on $|H_Q\rangle$ is the generator of the boost along direction \vec{v} . Indeed, to get $|H_Q(\vec{v})\rangle$ one must find the eigenstate of the Hamiltonian with heavy quark moving with the momentum $\vec{q}=m_Q \vec{v}$. The only part which explicitly depends on momentum comes from the heavy quark Hamiltonian $\frac{\vec{\pi}^2}{2m_Q}$ (plus, in general, the higher terms in $1/m_Q$). We use the relation $\exp(-i\vec{q}\vec{x}) \mathcal{H}_Q \exp(i\vec{q}\vec{x}) = \mathcal{H}_Q + \vec{v} \vec{\pi} + m_Q \vec{v}^2/2$ and drop the last term which is a constant; A_0 obviously commutes with x . Then Eq. (105) represents the first-order perturbation theory in $\delta\mathcal{H} = \vec{v} \vec{\pi}$; the unperturbed Hamiltonian is $\mathcal{H}_0 = \pi_0$ (further details can be found in [4], Eq. (178) and Sect. VI). In the second-quantized notations the very same relation takes the form

$$|H_Q(\vec{v})\rangle = |H_Q(0)\rangle + \int d^3\vec{x} \bar{Q} \pi_0^{-1} \vec{v} \vec{\pi} Q(x) |H_Q(0)\rangle + \mathcal{O}(\vec{v}^2). \quad (106)$$

The first four sum rules are obtained differently, using the OPE for the heavy quark transition operator. The Bjorken sum rule basically states that the probability of the quark to hadronize into some heavy flavor state after weak interaction, is exactly unity. In the different notations it was stated already in Ref. [45]. The

‘optical’ sum rule for $\bar{\Lambda}$ follows from the fact that no independent dimension-4 heavy quark operator exists for the forward matrix elements. This means that the apparent change in the kinematics when passing from the quark level to actual B mesons must be compensated in the average energy of the final states by the sums over excited states: the threshold for quark transition is at $q_0 = \frac{\vec{q}^2}{2m_Q}$ while for the hadron the elastic peak is at $q_0 = \frac{\vec{q}^2}{2M_{H_Q}}$, lower by $\bar{\Lambda} \frac{\vec{v}^2}{2}$. (The original derivation by Voloshin was different.)

The new sum rules (89) and (91) can be obtained applying the OPE to the nonforward scattering amplitude where the momenta flowing in and out the two weak vertices are not equal, so that the overall change in velocity $\vec{v} = \vec{v}_1 - \vec{v}_2$ is nonzero. They actually require the part antisymmetric in \vec{v}_1 and \vec{v}_2 .⁸ The new parameter of the heavy quark theory $\bar{\Sigma}$ is unfortunately unknown. It is defined as a small velocity elastic transition matrix element between the states with explicit spin of light degrees of freedom: for the ground-state vector mesons like B^*

$$\frac{1}{2M_{B^*}} \langle H_Q(\vec{v}, \varepsilon') | \bar{Q} i D_j Q(0) | H_Q(0, \varepsilon) \rangle = -\frac{\bar{\Lambda}}{2} v_j (\bar{\varepsilon}'^* \vec{\varepsilon}) + \frac{\bar{\Sigma}}{2} \{ (\bar{\varepsilon}'^* \vec{v}) \varepsilon_j - \varepsilon_j' (\bar{\varepsilon} \vec{v}) \} + \mathcal{O}(\vec{v}^2). \quad (107)$$

Alternatively, for spinless heavy quarks Q this would read as

$$\langle \Omega_0(\vec{v}) | \bar{Q} i D_j Q(0) | \Omega(0) \rangle = -\frac{\bar{\Lambda}}{2} v_j \Psi_0^\dagger \Psi_0 - i \frac{\bar{\Sigma}}{2} \epsilon_{jkl} v_k \Psi_0^\dagger \sigma_l \Psi_0 + \mathcal{O}(\vec{v}^2), \quad (108)$$

etc. Let us note that the full (quasielastic) matrix element in the l.h.s. of Eq. (107) is not well defined due to ultraviolet divergences in the static theory, and therefore the symmetric part proportional to $\bar{\Lambda}$ depends on regularization. Yet the antisymmetric part proportional to $\bar{\Sigma}$ is well defined and finite. It can be directly measured, for example, on the lattices. We expect $\bar{\Sigma}$ to be about 0.25 GeV.

Let us briefly outline the unified derivation of the sum rules Eqs. (88)-(91). The OPE approach leading to the sum rules is described in Sect. 5. Here, however, we consider the nonforward scattering amplitude similar to Eq. (87) with $J_\mu = J_\nu^\dagger = \bar{Q} \gamma_0 Q$ assuming that the final hadron has nonzero momentum $M_{H_Q} \vec{v}$:

$$T(q_0; \vec{v}, \vec{v}) = \frac{1}{2M_{H_Q}} \langle H_Q(\vec{v}) | \int d^3x dx_0 e^{i\vec{q}\vec{x} - iq_0 x_0} iT \{ J_0(x), J_0^\dagger(0) \} | H_Q(0) \rangle \quad (109)$$

with $\vec{q} \equiv m_Q \vec{v}$, and we retain only terms through second order in \vec{v} and \vec{v} . All terms suppressed by powers of $1/m_Q$ are discarded. As the energy variable we take $\epsilon = q_0 - (\sqrt{\vec{q}^2 + m_Q^2} - m_Q) \simeq q_0 - \frac{m_Q \vec{v}^2}{2}$. With this choice $\epsilon=0$ would correspond to

⁸This complication is not necessary for the sum rules for operators dimension 5 and higher (μ_G^2 , ρ_{LS}^3 , *etc.*) where the antisymmetric structure functions appear, for example, in B^* for the spacelike vector current vertices, although they are $1/m_Q^2$ suppressed. However, the excited states contribute to these structure functions with extra factor of ϵ^2 , and thus yield only the higher sum rules for local operators.

the elastic transition for free heavy quark; the larger ϵ , the larger is the mass of the intermediate state.

As explained in Sect. 5, we expand the amplitude Eq.(109) in $1/\epsilon$ assuming $m_Q \gg \epsilon \gg \Lambda_{\text{QCD}}$; the dispersion relation equates the terms in the $1/\epsilon$ expansion with the moments of the discontinuity of T due to on-shell transitions into intermediate states. The latter to the second order in velocities are the ground state ($B^{(*)}$) and the P -wave excitations. The OPE yields

$$\begin{aligned} -T(\epsilon; \vec{v}, \vec{v}) &= \frac{1}{\epsilon} \frac{1}{2M_{H_Q}} \langle H_Q(\vec{v}) | \bar{Q}(1 - \frac{\vec{v}^2}{4} + \frac{\vec{v}\vec{\gamma}}{2}) Q(0) | H_Q(0) \rangle \\ &+ \frac{1}{\epsilon^2} \frac{1}{2M_{H_Q}} \langle H_Q(\vec{v}) | \bar{Q}(i\vec{D}\vec{v}) Q(0) | H_Q(0) \rangle \\ &+ \frac{1}{\epsilon^3} \frac{1}{2M_{H_Q}} \langle H_Q(\vec{v}) | \bar{Q}[(i\vec{D}\vec{v})^2 - iD_0(i\vec{D}\vec{v})] Q(0) | H_Q(0) \rangle + \dots \end{aligned} \quad (110)$$

Comparing this to the explicit hadronic contributions to the small velocity moments $1/2\pi \int d\epsilon \epsilon^k \text{Im} T(\epsilon; \vec{v}, \vec{v})$ we get the stated sum rules. The structures antisymmetric in \vec{v} and $\vec{v} - \vec{v}$ emerge if H_Q carries spin correlated with the spin of light cloud, and for B^* yield the new sum rules involving the difference between the $\frac{3}{2}$ - and $\frac{1}{2}$ -contributions. The quasielastic transitions do not contribute to this structure for actual spin- $\frac{1}{2}$ quarks.

The computation proceeds similarly and is even simpler for spinless heavy quarks. The difference is only for the zeroth moment, sum rule (89): for scalar Q the anti-symmetric part is absent from the OPE expression for I_0 , but the elastic contribution yields it with the opposite sign.

The fact that the symmetric part of the $D = 4$ nonforward elastic matrix element (the first term in the r.h.s. of Eqs. (107)-(108)) is given by $\frac{\bar{\Lambda}}{2}$ follows from equations of motion:

$$\begin{aligned} M_{H_Q}(v-u)_\mu v_\mu \langle H_Q(\vec{v}) | \bar{Q}Q(0) | H_Q(0) \rangle &= -(iD_\mu v_\mu) \langle H_Q(\vec{v}) | \bar{Q}Q(x) | H_Q(0) \rangle \Big|_{x=0} = \\ &\langle H_Q(\vec{v}) | m_Q \bar{Q}Q(0) - \bar{Q}(iD_\mu v_\mu) Q(0) | H_Q(0) \rangle \end{aligned}$$

($u_\mu = (1, 0, 0, 0)$ is the restframe four-velocity and Q are the original full QCD fields), or, to order \vec{v}^2

$$(M_{H_Q} - m_Q) \frac{\vec{v}^2}{2} \langle H_Q | \bar{Q}Q(0) | H_Q \rangle = -v_k \langle H_Q(\vec{v}) | \bar{Q}iD_k Q(0) | H_Q(0) \rangle \quad (111)$$

which fixes the symmetric term in the above matrix elements.

The sum rule Eq. (89) deserves a special note. The sum of the differences between $\tau_{3/2}^2$ and $\tau_{1/2}^2$ amounts to a nonvanishing number which does not depend on strong dynamics. This can naturally raise a question of how this can be true, for the $\frac{3}{2}$ and $\frac{1}{2}$ states are differentiated only by spin-orbital interaction. For example, the light quark in the meson can be, in principle, arbitrary nonrelativistic as well, in which limit this interaction can be switched off.

The point is that in the nonrelativistic case τ^2 are large scaling like inverse square of the typical velocity of the light quark, $\tau^2 \sim 1/\bar{v}_{\text{sp}}^2$, and the relativistic spin-orbital effects must appear at the relative level $\sim \bar{v}_{\text{sp}}^2$ simply because spin ceases to commute with momentum to this accuracy. This then leads to corrections of order 1 in the first two sum rules Eqs. (88), (89). In the Bjorken sum rule the relativistic constant 1/4 is obscured by perturbative effects, while the sum rule Eq. (89) shows this universal relativistic effect in the most clean way.

As is seen from the preceding analysis, physics of the SV transitions is described by two “structure functions” W_+ and W_- :

$$\begin{aligned} W_+(\epsilon, \vec{v}) &= 2\bar{v}^2 \sum_m |\tau_{3/2}^{(m)}|^2 \delta(\epsilon - \epsilon_m) + \bar{v}^2 \sum_n |\tau_{1/2}^{(n)}|^2 \delta(\epsilon - \epsilon_n) + \xi(\bar{v}^2) \delta(\epsilon + \frac{\bar{\Lambda}\bar{v}^2}{2}), \\ W_-(\epsilon) &= 2 \sum_m |\tau_{3/2}^{(m)}|^2 \delta(\epsilon - \epsilon_m) - 2 \sum_n |\tau_{1/2}^{(n)}|^2 \delta(\epsilon - \epsilon_n) \end{aligned} \quad (112)$$

and the corresponding invariant hadronic tensors

$$\begin{aligned} h_+(\epsilon, \vec{v}) &= 2\bar{v}^2 \sum_m \frac{|\tau_{3/2}^{(m)}|^2}{\epsilon_m - \epsilon} + \bar{v}^2 \sum_n \frac{|\tau_{1/2}^{(n)}|^2}{\epsilon_n - \epsilon} - \frac{\xi(\bar{v}^2)}{\epsilon + \frac{\bar{\Lambda}\bar{v}^2}{2}}, \\ h_-(\epsilon) &= 2 \sum_m \frac{|\tau_{3/2}^{(m)}|^2}{\epsilon_m - \epsilon} - 2 \sum_n \frac{|\tau_{1/2}^{(n)}|^2}{\epsilon_n - \epsilon}. \end{aligned} \quad (113)$$

They directly enter, for example, $1/m$ corrections to various weak currents at zero recoil. Sum rules Eqs. (89) and (91) represent the exact low-energy theorems for $h_-(\epsilon)$.

Let us note that the sum rule (89) provides the rationale for the fact that pseudoscalar mesons B , D are lighter than their hyperfine partners B^* , D^* . If the sum rule for μ_G^2 is dominated by the low-lying states then μ_G^2 must be of the same sign as the constant in Eq. (89), which dictates the sign of the hyperfine mass splitting.

The sum rules (88)–(94) obviously entail a set of exact QCD inequalities. They are similar to those which have been with us since the early eighties [46], and reflect the most general features of QCD (like the vector-like nature of the quark-gluon interaction). The advent of the heavy quark theory paved the way to a totally new class of inequalities among the fundamental parameters. As with the old ones, they are based on the equations of motion of QCD and certain positivity properties. All technical details of the derivation are different, however, as well as the sphere of applications. The first in the series is the Bjorken inequality $\varrho^2 \geq 1/4$ [41]. We, in fact, have a stronger bound $\rho^2 \geq 3/4$. Other bounds include $\mu_\pi^2 \geq \mu_G^2$ and $\rho_D^3 \geq |\rho_{LS}^3|/2$, $\rho_D^3 \geq -\rho_{LS}^3$ and $\bar{\Lambda} \geq 2\bar{\Sigma}$ which are valid at arbitrary normalization point. We also have the direct inequalities between the parameters of different dimension:

$$\mu_\pi^2 \geq \frac{3\bar{\Lambda}^2}{4\varrho^2 - 1}, \quad \rho_D^3 \geq \frac{3}{8} \frac{\bar{\Lambda}^3}{(\varrho^2 - 1/4)^2}, \quad \rho_D^3 \geq \frac{(\mu_\pi^2)^{3/2}}{\sqrt{3(\varrho^2 - 1/4)}}. \quad (114)$$

All these inequalities are saturated if only one excited state contributes. A number of additional inequalities involving the energy of the first P -wave excitation Δ_1 is discussed in Ref. [21]. For further reference, let us also write an obvious consequence of the two equations for μ_π^2 and μ_G^2 :

$$\mu_\pi^2 - \mu_G^2 = 9 \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2. \quad (115)$$

4.2 Hard QCD and normalization point dependence

The sum rules (88)–(94) express heavy quark parameters, including $\bar{\Lambda} = M_B - m_b$, μ_π^2 and μ_G^2 as the sum of observable quantities, products of the hadron mass differences and transition probabilities. The observable quantities are scale-independent. How then, say, $\bar{\Lambda} = M_B - m_b$, μ_π^2 and μ_G^2 happen to be μ -dependent?

The answer is that in actual quantum field theory like QCD the sums over excited states are generally ultraviolet divergent when $\epsilon_k \gg \Lambda_{\text{QCD}}$; in contrast to ordinary quantum mechanics they are not saturated by a few lowest states with contributions fading out fast in magnitude with the excitation number. The contributions of hadronic states with $\epsilon_k \gg \Lambda_{\text{QCD}}$ are dual to what we calculate in perturbation theory using its basic objects, quarks and gluons. The latter yield the continuous spectrum and can be evaluated perturbatively using isolated quasifree heavy quarks as the initial state. The final states are heavy quarks and a certain number of gluons and light quarks. It is the difference between the actual hadronic and quark-gluon transitions that resides at low excitation energies.

Therefore, in order to make the sum rules meaningful, we must cut off the sums at some energy μ which then makes the expectation values μ -dependent. The simplest way is merely to extend the sum only up to $\epsilon_k < \mu$, and this is the convention we normally use. Thus, in reality all the sums in the relations (88)–(94) must include the condition $\epsilon_k < \mu$, which we omitted there for the sake of simplicity, and all the heavy quark parameters are normalized at the scale μ . The exception are superconvergent spin-nonsinglet sum rules (89) and (91) where in the perturbative domain $\mu \ll \Lambda_{\text{QCD}}$ such μ -dependence is power suppressed by factors $\alpha_s \Lambda_{\text{QCD}}/\mu$ and can be neglected. For analytic computations it is often convenient to apply the exponential cutoff factor $e^{-\epsilon_k/\mu}$, which is essentially the Borel transform of the related correlation functions. Since at large μ the cutoff factors differ only in the perturbative domain, the difference between various renormalization schemes can be calculated perturbatively.

The high-energy tail of the transitions to order α_s is given by the quark diagrams in Figs. 5 with

$$2 \sum_m \dots + \sum_n \dots \rightarrow \int \frac{d^3 \vec{k}}{2\omega}$$

where (ω, \vec{k}) is the momentum of the real gluon. The spin-singlet amplitudes are just a constant proportional to g_s , and performing the simple calculations we arrive

at the first-order term in the evolution of $\mu_\pi^2(\mu)$:

$$\frac{d\mu_\pi^2(\mu)}{d\mu^2} = \frac{4}{3} \frac{\alpha_s}{\pi} + \dots \quad (116)$$

Purely perturbatively, the continuum analogies of $\tau_{1/2}$ and $\tau_{3/2}$ are equal and a similar additive renormalization of μ_G^2 and ρ_{LS}^3 is absent.

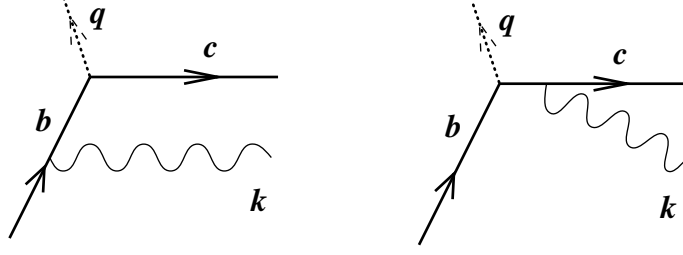


Figure 5: Perturbative diagrams determining the high-energy asymptotics of the heavy quark transition amplitudes and renormalization of the local operators.

The perturbatively obtained evolution equations (116), (72) allow one to determine the asymptotic values of $\tau_{1/2}$ and $\tau_{3/2}$ at $\epsilon \gg \Lambda_{\text{QCD}}$:

$$2 \sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 + \sum_n \epsilon_n^2 |\tau_{1/2}^{(m)}|^2 \rightarrow \frac{8}{9} \frac{\alpha_s(\epsilon)}{\pi} \epsilon d\epsilon, \quad (117)$$

$$\sum_m \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 - \sum_n \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 \rightarrow -\frac{3\alpha_s(\epsilon)}{2\pi} \frac{d\epsilon}{\epsilon} \left\{ \sum_{\epsilon_m < \epsilon} \epsilon_m^2 |\tau_{3/2}^{(m)}|^2 - \sum_{\epsilon_n < \epsilon} \epsilon_n^2 |\tau_{1/2}^{(n)}|^2 \right\}. \quad (118)$$

The last bracket is simply $\frac{1}{6}\mu_G^2(\epsilon)$. Eq. (117) can be extended to higher orders in α_s , this simply reduces to using the dipole coupling $\alpha_s^{(d)}(\epsilon)$ discussed in Sect. 2.4.1:

$$\alpha_s^{(d)}(\epsilon) = \bar{\alpha}_s \left(e^{-5/3 + \ln 2} \epsilon \right) - \left(\frac{\pi^2}{2} - \frac{13}{4} \right) \frac{\alpha_s^2}{\pi} + \mathcal{O}(\alpha_s^3). \quad (119)$$

($\bar{\alpha}_s$ is the standard $\overline{\text{MS}}$ strong coupling). Therefore we get a number of exact perturbative evolution equations [10]

$$\mu \frac{d\rho^2(\mu)}{d\mu} = \frac{8}{9} \frac{\alpha_s^{(d)}(\mu)}{\pi} \quad (120)$$

$$\frac{d\bar{\Lambda}(\mu)}{d\mu} = \frac{16}{9} \frac{\alpha_s^{(d)}(\mu)}{\pi} \quad (121)$$

$$\frac{d\mu_\pi^2(\mu)}{d\mu} = \frac{8}{3} \frac{\alpha_s^{(d)}(\mu)}{\pi} \mu \quad (122)$$

(the evolution for the Darwin operator is already more complicated since includes the usual anomalous dimension [47]). The similar questions for the alternative combination of the oscillator strengths entering μ_G^2 and ρ_{LS}^3 has not been studied yet.

4.3 On the saturation of the sum rules

The question of the saturation of the heavy quark sum rules (in particular, the lower ones Eqs. (88)-(93)) is of primary importance for phenomenology of the heavy quark expansion. The sum rules state what is the asymptotic value in the r.h.s. at large enough normalization point μ , and perturbation theory tells us their μ dependence. From which scale μ_0 this behavior applies, is a dynamical question. For superconvergent sum rules (89) and (91) this is the question at which scale the sums approach the stated values with a reasonable accuracy. In order to sensibly apply quantitative $1/m_Q$ expansion, one must have $m_Q > \mu_0$, possibly, $m_Q \gg \mu_0$. While this is probably the case for b particles, such a hierarchy is not obvious *a priori* in charm.

The existing numerical evaluations of $\bar{\Lambda}$ and μ_π^2 at the scale near 1 GeV suggest rather large values, approximately 0.7 GeV and 0.6 GeV², respectively, which impose rather tight constraints. These facts are often neglected under various pretexts, including challenging the accuracy of the QCD sum rules evaluations of μ_π^2 . The significant uncertainties, in principle, cannot be excluded. In a certain respect determination of $M_B - m_b$ from $e^+e^- \rightarrow b\bar{b}$ is somewhat indirect as well including mild theoretical assumptions. Indeed, even though QCD ensures that the quark mass appearing in the analysis of B decays and $b\bar{b}$ annihilation is the same, extrapolating from the decay (scattering) kinematics appropriate for B decays with small negative q^2 for the $b \rightarrow b$ transitions, to the timelike domain $q^2 \simeq 4m_b^2$ in the annihilation channel involves the long path. It brings in significant perturbative effects which are to be accounted for. If actual dynamics of gluonic degrees of freedom sets in to the perturbative regime late in energy, the associated uncertainties may, in principle, increase.

Nevertheless, we point out that certain constraints following from the sum rules can be confined within the world with a single heavy quark, and they still are tight. Namely, the value of $\mu_G^2 \simeq 0.4 \text{ GeV}^2$ has barely been challenged as extracted almost directly from $B^{(*)}$ and $D^{(*)}$ masses. By virtue of the sum rules, the value of μ_π^2 is at least as large. Thus, regardless of the accuracy in evaluations of the kinetic expectation value, the question can be phrased in terms of the generally accepted value of μ_G^2 . At which minimal scale μ_0 the value of $\mu_G^2(\mu_0)$ reaches 0.3 or 0.4 GeV²? If this scale is below 1 GeV, large $\bar{\Lambda}$ and μ_π^2 are almost inevitable. If, however, $\mu_G^2(1 \text{ GeV})$ is significantly below 0.4 GeV², the chances for success in $1/m_Q$ expansion in charm are slim.

The estimates for $\tau_{3/2}$ and $\tau_{1/2}$ for the lowest P wave states group around 0.4, with $\epsilon_{3/2}^{(1)} \gtrsim \epsilon_{1/2}^{(1)} \simeq 400$ to 500 MeV (for the review see [48]). Similar values were reportedly extracted from the overall experimental yield of the corresponding charmed P wave states [49]. It is evident that such oscillator strengths fall short in saturating the sum rules:

$$\delta_{3/2}^{(1)} \mu_G^2 \simeq 0.2 \text{ GeV}^2, \quad \delta_{3/2}^{(1)} \bar{\Lambda} \simeq 0.3 \text{ GeV}.$$

In principle, they alone should not necessarily, even though the idea of the dominance of the lowest states contribution is very appealing. Let us mention that in the

't Hooft model all the heavy quark sum rules are saturated with amazing accuracy by the first excitations [50, 51]. Is such a possibility excluded in QCD? Probably, not completely. The dominance of the first excitation with $\epsilon \simeq 500$ MeV (recall that one must use the asymptotic $m_Q \rightarrow \infty$ values of the excitation energies and amplitudes) is still possible if QCD sum rules underestimate the value of $\tau_{3/2}^{(1)}$.⁹ Experimental determinations of τ 's are also questionable since $1/m_c$ corrections are not accounted for there. The estimates in the 't Hooft model suggest that they can be very large. In the cases when they are known explicitly in QCD, the $1/m_c$ terms generally turn out to be very significant as well [52].

Another – and, apparently, the most natural – option is that there are new states with the masses around 700 MeV with similar, or even larger $\tau_{3/2}^{(2)} \simeq 0.4$ to 0.5; they can be broad and not identified with clear-cut resonances. The $\frac{1}{2}$ -states must be yet depleted up to this scale. All such states can be produced in semileptonic b decays and observed as populating the domain of hadronic invariant mass below or around 3 GeV. It will be important to explore these questions in experiment.

The similar constraints and the pattern of saturation follow from the sum rule (91) and, in particular, (89). The most natural “favorable” solution would imply existence of $\frac{3}{2}$ states around $\epsilon \approx 700$ MeV with significant $\tau_{3/2} \approx 0.5$, and suppressed transition amplitudes for the $\frac{1}{2}$ states in the whole domain of ϵ below 1 GeV.

5 Heavy Flavor Sum Rules; Finite m_Q

In practical applications we usually need to know the decay amplitudes for actual b particles; therefore, the mass of the initial heavy quark, although large compared to Λ_{QCD} , is finite and the corresponding corrections must be taken into account. The c quark in the final state is only marginally heavy, and these corrections are typically very significant. In $b \rightarrow u$ transitions the final state quark is light and the literal $1/m_Q$ expansion becomes inapplicable. However, in certain cases – for example, in the sum rules – the energy of the final hadronic state rather than m_q itself is an expansion parameter, and one still can use the similar expansion as long as the transferred spacelike momentum is large [4].

The finite- m_Q sum rules are less compact for a number of reasons. First, one needs to account for the various $1/m_Q$ -suppressed terms; yet the corrections are expressed in terms of the hadronic parameters we have encountered already analyzing the static SV sum rules. This is true if one expresses everything in terms of the quark-level kinematics. Since in practice we count all the energies from the observable hadronic thresholds, the nonlocal correlators which determine the expansion of the heavy hadron mass in powers of $1/m_Q$ enter as well. This is the only way how, for example, $\bar{\Lambda}$ enters the sum rules. Another source of nonlocal correlators of heavy

⁹Let us note that the technology of the QCD sum rules assumes the approximate duality starting $\epsilon = 1$ GeV or even lower (the energy are counted from the heavy quark mass there). Therefore accepting poor saturation of the exact heavy quark sum rules at this scale and still relying on the QCD sum rules predictions is not self-consistent.

quark operators is the deviation of the expectation values over the initial hadron (B meson) from their asymptotic values which would exist if $m_b \rightarrow \infty$.

Second, beyond the static approximation external weak currents proliferate, and as much as five independent structure functions describe the decays even for the standard $V-A$ current. Likewise, we often have to go beyond the SV approximation and do not assume that the final state velocity is small. Therefore, the structure functions as in usual deep inelastic scattering (DIS) become dependent on two kinematic variables, say q_0 and q^2 . However, to keep the parallel with the heavy quark and the SV limit it is often advantageous to choose instead q_0 and $|\vec{q}| = \sqrt{q_0^2 - q^2}$. (This choice is also convenient in a number of practical applications to $b \rightarrow c$ decays.)

The sum rules are derived in QCD using the standard methods of the short-distance expansion [4]. One starts the analysis from the forward transition amplitude

$$T^{(12)}(q_0; \vec{q}) = \frac{1}{2M_B} \int d^3\vec{x} dx_0 e^{i\vec{q}\vec{x} - iq_0 x_0} \langle B | iT \{ \bar{c} \Gamma^{(1)} b(x), \bar{b} \Gamma^{(2)} c(0) \} | B \rangle \quad (123)$$

where $\Gamma^{(1,2)}$ are some spin structures or, more generally, local operators. The transition amplitude contains a lot of information about the decay probabilities. As usual in QCD, one cannot calculate it completely in the physical domain of q . The amplitude (123) has several cuts corresponding to different physical processes. The discontinuity at the physical cut $q_0 < M_B - \sqrt{M_D^2 + \vec{q}^2}$ describes the inclusive decay probabilities at a given energy released into the final hadronic system. The cut continues further than the domain accessible in the actual decays, see Fig. 6.

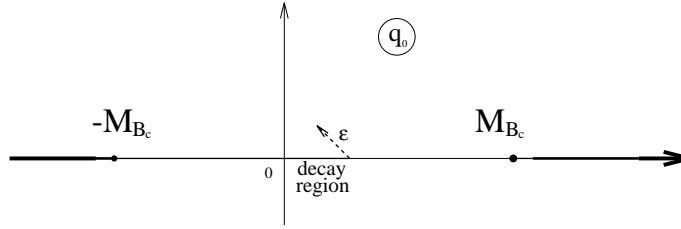


Figure 6: Cuts of the transition amplitude in the complex q_0 plane. The physical cut for the weak decay starts at $q_0 = M_B - (M_D^2 + \vec{q}^2)^{1/2}$ and continues towards $q_0 = -\infty$. Other physical processes generate cuts starting near $q_0 = \pm (M_{B_c}^2 + \vec{q}^2)^{1/2}$ (one pair); another pair of cuts originates at close values of q_0 . An additional channel opens at $q_0 \gtrsim 2m_b + m_c$.

For the practical case of left-handed currents $\Gamma^{(1)} = \gamma_\mu(1 - \gamma_5)$, $\Gamma^{(2)} = \gamma_\nu(1 - \gamma_5)$ and the hadronic tensor $T^{(12)}(q_0; \vec{q})$ can be decomposed into five covariants [53]

$$T_{\mu\nu} = -h_1 g_{\mu\nu} + h_2 v_\mu v_\nu - i h_3 \epsilon_{\mu\nu\alpha\beta} v_\alpha q_\beta + h_4 q_\mu q_\nu + h_5 (q_\mu v_\nu + q_\nu v_\mu), \quad (124)$$

each having the same analytic and unitarity properties. In particular, they obey the dispersion relation

$$h_i(q_0) = \frac{1}{2\pi} \int \frac{w_i(\tilde{q}_0) d\tilde{q}_0}{\tilde{q}_0 - q_0} + \text{polynomial}, \quad (125)$$

where $v_\mu = (1, 0, 0, 0)$ is the four-velocity of the decaying meson and w_i are observable structure functions:

$$w_i = 2 \operatorname{Im} h_i . \quad (126)$$

For decays into massless leptons the structure functions w_4 and w_5 do not contribute due to conservation of the lepton current; they are proportional to m_ℓ^2 and are relevant only for the decays into τ lepton [54].

In QCD we calculate the amplitude (123) away from all its cuts. Essentially, it can be expanded in the inverse powers of the distance from the physical cut, ϵ :

$$\epsilon = M_B - \sqrt{M_D^2 + \vec{q}^2} - q_0 = m_b - \sqrt{m_c^2 + \vec{q}^2} + \delta(\vec{q}^2) - q_0 , \quad (127)$$

where

$$\delta(\vec{q}^2) = (M_B - m_b) - \left(\sqrt{M_D^2 + \vec{q}^2} - \sqrt{m_c^2 + \vec{q}^2} \right) \sim \mathcal{O} \left(\Lambda_{\text{QCD}} \frac{\vec{q}^2}{m_Q^2} \right) + \mathcal{O} \left(\frac{\Lambda_{\text{QCD}}^2}{m_Q} \right) .$$

The “off-shellness” ϵ must be chosen in such a way that that $|\epsilon|$ and $|\epsilon \cdot \arg \epsilon| \gg \Lambda_{\text{QCD}}$ but, simultaneously, $|\epsilon| \ll \sqrt{m_c^2 + \vec{q}^2}$, m_b . The second requirement allows us to “resolve” the contributions of the separate cuts.

How does the deep Euclidean expansion of $T^{(12)}$ help to constrain the amplitude in the physical domain of real ϵ lying just on the cut? The dispersion relations Eq. (125) tells us that the coefficients in the expansion of the amplitude in powers of $1/\epsilon$ are given by the corresponding moments of the spectral density in $T^{(12)}$,

$$T^{(12)}(\epsilon; \vec{q}) = \frac{1}{\pi} \int \frac{\operatorname{Im} T^{(12)}(\epsilon'; \vec{q})}{\epsilon' - \epsilon} d\epsilon' = -\frac{1}{\pi} \sum_{k=0}^{\infty} \frac{1}{\epsilon^{k+1}} \int \operatorname{Im} T^{(12)}(\epsilon'; \vec{q}) \epsilon'^k d\epsilon' ; \quad (128)$$

the similar expansions hold for separate invariant structures.

On the other hand, we can build the large- ϵ expansion of the transition amplitude *per se*, treating it as the propagation of the virtual heavy quark submerged into a soft medium. The expansion takes the general form

$$\begin{aligned} T^{(12)}(q_0; \vec{q}) &= \frac{1}{2M_B} \int d^3\vec{x} dx_0 e^{i\vec{q}\vec{x} - iq_0 x_0} \times \\ &\langle B | \bar{b}(x) \Gamma^{(1)} \left[(m_c + i\not{D}) \frac{1}{m_c^2 - (iD)^2 - \frac{i}{2}\sigma G} \right]_{x_0} \Gamma^{(2)} b(0) | B \rangle = \\ &\langle \Gamma^{(1)} [m_c + m_b \gamma_0 - \not{q} + \not{\epsilon}] \frac{1}{m_c^2 - (m_b - q_0)^2 + \vec{q}^2 - 2m_b \pi_0 + 2q\pi - \pi^2 - \frac{i}{2}\sigma G} \Gamma^{(2)} \rangle = \\ &\langle \Gamma^{(1)} [m_c + (E_c + \epsilon - \delta(\vec{q}^2)) \gamma_0 + \vec{q} \vec{\gamma} + \not{\epsilon}] \times \\ &\sum_{n=0}^{\infty} \frac{[2(E_c + \epsilon - \delta(\vec{q}^2)) \pi_0 + 2\vec{q} \vec{\pi} + \pi^2 + \frac{i}{2}\sigma G]^n}{[-(\epsilon - \delta(\vec{q}^2))(2E_c + \epsilon - \delta(\vec{q}^2))]^{(n+1)}} \Gamma^{(2)} \rangle . \end{aligned} \quad (129)$$

Here $E_c = \sqrt{m_c^2 + \vec{q}^2}$, the symbol $[\dots]_{x0}$ denotes the $0x$ matrix element of the corresponding operator (it can be understood as the integral operator in the coordinate representation), and we use the short-hand notation

$$\langle \dots \rangle = \frac{1}{2M_B} \int d^3\vec{x} dx_0 \langle B | \bar{b}(x) [\dots]_{x0} b(0) | B \rangle .$$

In the first equation we used the full QCD fields and then passed to the low-energy fields according to Eqs. (12)–(14). Picking up the corresponding term $1/\epsilon^{k+1}$ in the expansion over $1/\epsilon$ at $\epsilon \gg \Lambda_{\text{QCD}}$ and evaluating the expectation value of the resulting *local* operators (e.g., $\bar{b}(0)\pi_\mu b(x) \equiv \delta^4(x)\bar{b}\pi_\mu b(0)$), one gets the sum rules sought for. Taking $k=0$ yields the sum rule for the equal-time commutator of the currents $\bar{c}\Gamma^{(1)}b$ and $\bar{b}\Gamma^{(2)}c$; $k=1$ selects the commutator for the time derivative, *etc.*

Applying this general procedure to the heavy quark transition amplitude, we face a subtlety which is specific to the OPE in heavy quarks. Namely, speaking formally the dispersion relation represents the amplitude as a Cauchy integral over *all* cuts it has, while we try to reconstruct the structure functions (or their moments) referring to a particular cut describing the heavy quark decay process. It was first discussed in Ref. [4]. Performing the OPE we identify the integrals over the separate cuts in the quark Green functions with the corresponding cuts in the physical amplitudes. This identification is established order by order in $1/m_Q$ expansion since the contributions of the physical cut behave as powers in $1/\epsilon$ while the contributions of the other, “distant” cuts behave like $1/(\epsilon+M)^n$ where $M \sim m_Q$ is the distance to the other cut, and therefore they are regular at $\epsilon \ll m_Q$. This property was given the name of *global duality* [4]. Beyond the practical version of the OPE global duality is, in general, violated and may play the role of additional theoretical background together with local duality violations.¹⁰

The problem of separating the extra cut contributions would become serious, for example, in the $b \rightarrow u$ transitions if $\sqrt{q^2}$ approaches m_b . The u -channel cut then becomes too close to the physical decay cut [55], and their contributions in the correlator Eq. (123) cannot be taken apart. This complication originates from the presence of the reverse ordering of the two weak decay vertices in the correlator. Naively, it could have been avoided if one considered the retarded – instead of the time-ordered – product in the correlation function, or simply defined the hadronic tensor as the dispersion integral only over the discontinuity of the decay cut (continued towards larger q_0 accessible, in principle, in the process of scattering of the weak current on the heavy hadron). However, in order to apply the technique of the OPE, we must have two necessary ingredients: first, the amplitude must have the proper analytic properties, and, second, be represented as the correlator of local field operators for which we can write the functional representation in the background gluon field. The combination of the two requirements dictates using the T -ordered product in Eq. (123), and thus makes the question of global duality unavoidable.

¹⁰It is probable, however, that global duality is not too sensitive to the peculiarities of Minkowskian kinematics, and its violations are truly exponentially suppressed at large m_Q .

We quote here the expressions for the first three moments of the structure functions $w_{1,2,3}$ for the standard $V-A$ weak currents [4]; the higher moments require accounting for the nonperturbative effects beyond $\mathcal{O}(\Lambda_{\text{QCD}}^2)$. The variable ϵ describing the excess of the final state hadronic energy above the threshold is defined as

$$\epsilon = M_B - \sqrt{M_{D^*}^2 + \vec{q}^2} - q_0, \quad (130)$$

i.e., is counted from the energy of the D^* meson (rather from the lowest state D contributing to the sum rules); this convention does not affect the zeroth moments. Moments of the structure functions are

$$I_n^{(i)}(\vec{q}^2) = \frac{1}{2\pi} \int d\epsilon \epsilon^n w_i(\epsilon, \vec{q}^2) \quad (131)$$

where i labels the structure function; they can be considered separately for axial current (AA), vector current (VV) and for the interference of the two (AV).

For the zeroth moments we have

$$I_0^{(1)AA} = \frac{E_c + m_c}{2E_c} - \frac{\mu_\pi^2 - \mu_G^2}{4E_c^2} \frac{m_c}{E_c} \left[\frac{m_c^2}{E_c^2} + \frac{E_c^2}{m_b^2} + \frac{2m_c}{3m_b} \right] - \frac{\mu_G^2}{3E_c^2} \frac{m_c}{E_c} \frac{E_c^2 + 3m_c^2}{4E_c^2} \quad (132)$$

$$I_0^{(2)AA} = \frac{m_b}{E_c} \left\{ 1 - \frac{\mu_\pi^2 - \mu_G^2}{3E_c^2} \left[2 - \frac{5E_c^2}{2m_b^2} + \frac{3m_c^2}{2E_c^2} \right] - \frac{\mu_G^2}{3E_c^2} \left[\frac{1}{2} + \frac{m_c}{m_b} + \frac{3m_c^2}{2E_c^2} \right] \right\} \quad (133)$$

$$I_0^{(3)AV} = -\frac{1}{2E_c} \left\{ 1 - \frac{\mu_\pi^2 - \mu_G^2}{3E_c^2} \left[1 + \frac{3m_c^2}{2E_c^2} \right] - \frac{\mu_G^2}{2E_c^2} \left[1 + \frac{m_c^2}{E_c^2} \right] \right\}. \quad (134)$$

Expressions for the VV functions are obtained from the axial ones by replacing $m_c \rightarrow -m_c$; the structure functions $w^{(1,2)AV}$ and $w^{(3)AA,VV}$ vanish. The above equations are analogies of the Bjorken sum rule; however they incorporate nonperturbative effects which appear at $1/m_Q^2$ level; the corrections are not universal and differ explicitly for different currents and structure functions.

The first moments look as follows:

$$I_1^{(1)AA} = \frac{E_c + m_c}{2E_c} \left\{ \frac{\mu_\pi^2 - \mu_G^2}{2E_c} \left[1 - \frac{E_c}{m_b} - \frac{1}{3} \left(1 - \frac{m_c}{E_c} \right) \left(1 + 3\frac{m_c}{E_c} + 2\frac{E_c}{m_b} \right) \right] + \right. \\ \left. + \frac{\mu_G^2}{2E_c} \left[1 - \frac{2m_c}{3E_c} + \frac{m_c^2}{E_c^2} \right] + [(M_B - m_b) - (E_{D^*} - E_c)] \right\} \quad (135)$$

$$I_1^{(2)AA} = \frac{m_b}{E_c} \left\{ \frac{\mu_\pi^2 - \mu_G^2}{3E_c} \left[2 - \frac{7E_c}{2m_b} + \frac{3m_c^2}{2E_c^2} \right] + \frac{\mu_G^2}{3E_c} \left[\frac{1}{2} - \frac{E_c - m_c}{m_b} + \frac{3m_c^2}{2E_c^2} \right] + \right. \\ \left. + [(M_B - m_b) - (E_{D^*} - E_c)] \right\} \quad (136)$$

$$I_1^{(3)AV} = -\frac{1}{2E_c} \left\{ \frac{\mu_\pi^2 - \mu_G^2}{3E_c} \left[1 - \frac{5E_c}{2m_b} + \frac{3m_c^2}{2E_c^2} \right] + \frac{\mu_G^2}{2E_c} \left[1 + \frac{m_c^2}{E_c^2} \right] + \right. \\ \left. + [(M_B - m_b) - (E_{D^*} - E_c)] \right\} \quad (137)$$

At $\vec{q} = 0$ these relations determine $1/m_Q$ terms in the masses of heavy mesons. Their derivatives with respect to \vec{q}^2 near zero recoil give the Voloshin's "optical" sum rule. Here they are obtained with better accuracy for arbitrary, not necessary small, velocity and incorporate $1/m_Q$ relative corrections. The latter appear to be quite sizable when \vec{q} is not particularly large.

The third sum rules, which are relations for the second moments of the structure functions, are calculated only in the leading non-trivial approximation. They look rather simple and manifestly satisfy the heavy quark symmetry relation [43]:

$$\begin{aligned} \frac{2E_c}{E_c+m_c} I_2^{(1)AA} &= \frac{2E_c}{E_c-m_c} I_2^{(1)VV} = \frac{E_c}{m_b} I_2^{(2)AA} = \frac{E_c}{m_b} I_2^{(2)VV} = -2E_c I_2^{(3)AV} = \\ &= \frac{\mu_\pi^2}{3} \frac{E_c^2 - m_c^2}{E_c^2} + \bar{\Lambda}^2 \left(1 - \frac{m_c}{E_c}\right)^2. \end{aligned} \quad (138)$$

All higher moments vanish in our approximation. We used above the notation E_{D^*} for the energy of D^* and E_c for the energy of the c quark in the free quark decay:

$$E_{D^*} = \sqrt{M_{D^*}^2 + \vec{q}^2}, \quad E_c = \sqrt{m_c^2 + \vec{q}^2}. \quad (139)$$

The quantity $(M_B - m_b) - (E_{D^*} - E_c)$ which enters first and second moments determines the difference in kinematics between the free quark and the quasielastic $B \rightarrow D^*$ transitions. At zero recoil $\vec{q}=0$ it is of order Λ_{QCD}^2 , however in the general situation scales like Λ_{QCD}^1 :

$$\begin{aligned} (M_B - m_b) - (E_{D^*} - E_c) &\simeq \\ \bar{\Lambda} \left(1 - \frac{m_c}{E_c}\right) - (\mu_\pi^2 - \mu_G^2) \left(\frac{1}{2E_c} - \frac{1}{2m_b}\right) - \frac{2\mu_G^2}{3E_c} - \frac{\bar{\Lambda}^2}{2E_c} \left(1 - \frac{m_c^2}{E_c^2}\right) &+ \mathcal{O}\left(\frac{1}{m^2}\right). \end{aligned} \quad (140)$$

The explicit expressions for the structure functions themselves including these nonperturbative corrections, and the semileptonic decay distributions are given in Ref. [53]. Pure perturbative corrections to the heavy quark structure functions were computed in Ref. [56].

5.1 Zero recoil sum rules; $|V_{cb}|$ from $B \rightarrow D^* \ell \nu$ at zero recoil.

5.1.1 Sum rules for μ_π^2 and μ_G^2

Considering the semileptonic transitions driven by the pseudoscalar weak current $J_5 = \int d^3\vec{x} \bar{c} i \gamma_5 b(x)$ (i.e., at zero recoil $\vec{q}=0$) one obtains the sum rule

$$\frac{1}{2\pi} \int_0^\mu w^{(5)}(\epsilon) d\epsilon = \sum_{\vec{\epsilon}_k < \mu} |\tilde{F}_k|^2 = \left(\frac{1}{2m_c} - \frac{1}{2m_b}\right)^2 (\mu_\pi^2(\mu) - \mu_G^2(\mu)) \quad (141)$$

yielding the inequality $\mu_\pi^2(\mu) \geq \mu_G^2(\mu)$; this is the field-theoretic analogue of the relation Eq. (115). This correspondence is transparent keeping in mind the nonrelativistic expansion of the pseudoscalar current $\bar{c} i \gamma_5 b \simeq (1/2m_c - 1/2m_b) \varphi_c^+ \vec{\sigma} \vec{\pi} \varphi_b$ at

zero momentum transfer. Only the transitions to the P -wave states survive here in the leading in $1/m_Q$ approximation.

The sum rule (102) for μ_G^2 can also be readily obtained in this way. For example, one can consider the antisymmetric (with respect to i and j) part of the correlator of the vector currents ($\Gamma^{(1)} = \gamma_i$, $\Gamma^{(2)} = \gamma_j$); such a sum rule must be considered for B^* mesons (in B the expectation value of \vec{B}_{chr} vanishes). Alternatively, the sum rule for the correlator of the nonrelativistic currents $\bar{c}\pi_j b$ and $\bar{b}\vec{\sigma}\pi c$ directly can be considered. The OPE guarantees that all such relations are equivalent.

5.1.2 F_{D^*} at zero recoil

The concept of the heavy quark symmetry was very important for the evolution of studies of heavy flavor hadrons. The fact of the fixed normalization of the $B \rightarrow D$ and $B \rightarrow D^*$ formfactors at zero recoil in the limit $m_{b,c} \rightarrow \infty$ was of special significance in applications since suggested the method for determination of $|V_{cb}|$ from the $B \rightarrow D^*$ semileptonic decay channel near zero recoil. To this end one measures its differential rate, extrapolates to the point of zero recoil and gets the quantity $|V_{cb} F_{D^*}(0)|$, where F_{D^*} is the axial $B \rightarrow D^* \ell \nu$ formfactor. Since the charm quark is only marginally heavy, it is very important to estimate the corrections, in particular, nonperturbative. The exclusive transition amplitudes are not genuinely short-distance, such transitions proceed in time intervals $\sim 1/\Lambda_{\text{QCD}}$. Nevertheless, it turns out that the large-distance effects appear in these kinematics only suppressed by $1/m_{c,b}^2$:

$$F_{D^*}(0) = 1 + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right) + \delta_{1/m^2}^A + \delta_{1/m^3}^A + \dots \quad (142)$$

The absence of $1/m_Q$ corrections was first noted in Ref. [45].¹¹ It can be readily understood. Let us consider, for example, the vector $B \rightarrow D$ transition at zero recoil:

$$\langle D | \bar{c} \gamma_0 b | B \rangle = 2\sqrt{M_B M_D} \left\{ 1 + \frac{a}{m_c} - \frac{a}{m_b} + \dots \right\} \quad (143)$$

(short-distance effects are neglected). The relative magnitude of $1/m_b$ and $1/m_c$ terms is fixed since at $m_b = m_c$ all corrections must vanish identically. T -invariance, however, says that the coefficients for $1/m_c$ and $1/m_b$ terms must be equal since B differs from D by only the value of the heavy quark mass. Thus, both terms must vanish. Actually, this statement is the heavy-quark analogue of the Ademollo-Gatto theorem for the $SU(3)_{\text{fl}}$ breaking effects which is routinely exploited in determinations of $|V_{us}|$ from $K \rightarrow \pi \ell \nu$ and semileptonic hyperon decays. This observation was later studied in more detail in Ref. [6] and is usually called Luke's theorem. It improves the credibility of this method of determination of $|V_{cb}|$ in spite of certain experimental difficulties.

Since the corrections to the heavy quark limit are governed by the mass of the charm quark, even the $1/m_Q^2$ effects *a priori* can be significant. The need in

¹¹The first preprint version of the paper [57] discussing the heavy quark symmetry in QCD was given to me by M. Shifman in July 1986.

evaluation of the $1/m_Q^2$ corrections in Eq. (142) for practical purposes was realized quite early [58]. In these days the theory of the power corrections in heavy quarks was immature, so that it was hard to decide even the sign of δ_{1/m^2}^A . Quantitative application of the heavy quark symmetry to charm was generally viewed overly optimistic; it was believed that the deviations from the symmetry limit in F_{D^*} must be very small, at the scale of 2%, and suggestions that they could be as large as 10% were categorically refuted [58]. At present, with the application of dynamic methods in the heavy quark expansion, the actual estimates of F_{D^*} rather fall close to 0.9 [9].

The existing estimates of the power nonperturbative corrections in F_{D^*} are based on the sum rules for heavy flavor transition amplitude discussed in the previous section. Studying the zero-recoil transition we fix the spacelike momentum $\vec{q}=0$. The axial current $\bar{c}\gamma_i\gamma_5 b$ produces D^* , $D\pi$ and higher excitations in semileptonic B decays at zero recoil. A straightforward derivation yields the following sum rule for this current (cf. Eq. (132) for $3h_1^{AA}$ at $\vec{q}=0$):

$$|F_{D^*}|^2 + \sum_{0 < \epsilon_i < \mu} |F_i|^2 = \xi_A(\mu) - \Delta_{1/m^2}^A - \Delta_{1/m^3}^A + \mathcal{O}\left(\frac{1}{m_Q^4}\right), \quad (144)$$

where

$$\Delta_{1/m^2}^A = \frac{\mu_G^2(\mu)}{3m_c^2} + \frac{\mu_\pi^2(\mu) - \mu_G^2(\mu)}{4} \left(\frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_c m_b} \right) \quad (145)$$

(the $1/m_Q^3$ corrections to the sum rule Δ_{1/m^3}^A are also known, Eq. (8.21) of Ref. [21]), and the elastic formfactor F_{D^*} is defined as $\langle D^*(\vec{q}=0) | \bar{c}\gamma_i\gamma_5 b | B \rangle = \sqrt{2M_B M_{D^*}} F_{D^*} \epsilon_i^*$. F_i denote the axial-current transition formfactors to excited charm states i with the mass $M_i = M_{D^*} + \epsilon_i$, and ξ_A is a short-distance renormalization factor. Contributions from excitations with ϵ higher than μ are dual to perturbative contributions and get lumped into the coefficient $\xi_A(\mu)$ of the unit operator, the first term in the right-hand side of Eq. (144).

The role of μ is thus two-fold: in the left-hand side it acts as an ultraviolet cutoff in the effective low-energy theory, and by the same token determines the normalization point for the local operators; simultaneously, it defines the infrared cutoff in the Wilson coefficients.

The sum rule Eq. (144) leads to the upper bound:

$$|F_{D^*}|^2 \simeq \xi_A(\mu) - \Delta_{1/m^2}^A - \Delta_{1/m^3}^A - \sum_{0 < \epsilon_i < \mu} |F_i|^2, \\ -\delta_{1/m^2}^A > \frac{1}{2}\Delta_{1/m^2}^A \geq \frac{M_{B^*}^2 - M_B^2}{8m_c^2} \simeq 0.035. \quad (146)$$

The last relation is a model-independent lower bound for the $1/m^2$ corrections to F_{D^*} at zero recoil [9].

To obtain an actual estimate of the nonperturbative corrections rather than a bound, we need to know something about the contribution of the excited states in the

sum rule Eq. (144). Unfortunately, no model-independent answer to this question exists at present. The best we can do is to assume that the sum over the excited states is a fraction χ of the local term given by μ_π^2 and μ_G^2 ,

$$\sum_{\epsilon_i < \mu} |F_i|^2 = \chi \Delta_{1/m^2}^A, \quad (147)$$

where on general grounds $\chi \sim 1$. The contribution of the continuum $D\pi$ state can be calculated [9], however theoretically it is expected to constitute only a small fraction of the sum over resonant states. Trying to be optimistic, we rather arbitrarily limit χ by unity on the upper side, that is, put $\chi = 0.5 \pm 0.5$ [9]; the larger is χ , the smaller is F_{D^*} .

Arguments in favor of the assumption $\chi \leq 1$ are rather soft. In perturbation theory $\chi = 1$ holds in the first order (see [4], Sect. VII), but this relation changes in higher orders. The fraction χ was recently computed analytically [51] in the exactly solvable 't Hooft model (QCD in (1+1) dimensions in the limit of large number of colors) and turned out to be close to 0.55; in this model it is almost saturated by the first radial excitation with $\epsilon \approx 700$ MeV.

The short-distance renormalization factor $\xi_A(\mu)$ is calculated perturbatively. To the first order it was computed in [4], all-order BLM resummation performed in [23]. The most technically involved part of genuine $\mathcal{O}(\alpha_s^2)$ corrections to $\xi_A(\mu)$ was computed in [59], and the complete result given in [60]; the non-BLM corrections turn out to be small slightly decreasing $\xi_A(\mu)$, with the overall estimate $\xi_A(\mu) \simeq (0.99)^2$ at μ around 0.7 GeV. These computations, however were accomplished in expansion in μ/m_Q and the terms to order μ^2/m_Q^2 were retained, which is consistent if only $1/m_Q^2$ correction to the formfactor F_{D^*} are addressed. In view of the presence of terms suppressed by powers of $1/m_c$, the higher-order terms are expected to be significant. This was suggested by the perturbative BLM resummation itself where, in particular, $1/m_c^3$ corrections in $\xi_A^{1/2}$ were found to be at least at the scale of 2%. To estimate such effects, it is possible to evaluate $\xi_A(\mu)$ completely as a function of μ/m_Q , in the BLM approximation which presumably yields the dominant contribution. The typical numerical outcome of such computations is shown in Fig. 7. It is thus reasonable to accept the value $\xi_A^{1/2}(\mu) \simeq 0.96 \pm 0.015$ at $\mu \approx 0.8$ GeV for the short-distance renormalization of the axial current.¹² The uncertainty here in the short-distance renormalization can hardly be significantly reduced further since the inherent momentum scale is relatively low. The $\mathcal{O}(\alpha_s)$ corrections to the Wilson coefficient of the kinetic operator in the sum rule Eq. (145) was also computed to the next-to-leading order in [60]; the correction turns out to be insignificant.

¹²A similar number, with much smaller uncertainty is often cited in the literature for the so-called η_A factor introduced in HQET to denote ‘purely perturbative’ scale-independent renormalization of the axial current at zero recoil. Technically η_A coincides with $\xi_A^{1/2}(\mu)$ at $\mu \rightarrow 0$ to any order in the perturbative expansion. Regardless of intrinsic deficiencies of such a notion, the numerical value of η_A has the irreducible infrared renormalon uncertainty of at least 3-5% which by itself is much larger than the overall uncertainty in η_A quoted in the literature, see, *e.g.* [61]. The numerical close coincidence of the value quoted for η_A and $\xi_A^{1/2}(0.8 \text{ GeV})$ is accidental.

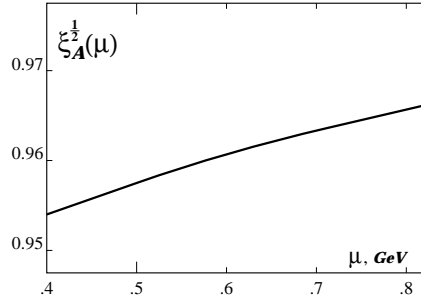


Figure 7: Short-distance renormalization $\xi_A^{\frac{1}{2}}(\mu)$ of the zero recoil $b \rightarrow c$ axial current as a function of μ . $\Lambda_{\text{QCD}} = 250 \text{ MeV}$.

The $1/m_Q^3$ term in the zero recoil sum rule Eq. (144) $-\Delta_{1/m^3}^A$ is estimated to constitute about -3% . Although it further suppresses the apparent value of F_{D^*} , we discard it in our estimates. It is expected to be only a smaller fraction of the overall $1/m_Q^3$ effects in the $B \rightarrow D^*$ formfactor, and keeping Δ_{1/m^3}^A is hardly legitimate since the leading correction δ_{1/m^2}^A in the formfactor is not known completely.

Assembling all pieces together we get for $\chi = 0.5 \pm 0.5$

$$F_{D^*} \simeq 0.89 - 0.015 \frac{\mu_\pi^2 - 0.5 \text{ GeV}^2}{0.1 \text{ GeV}^2} \pm 0.025_{\text{excit}} \pm 0.015_{\text{pert}} \pm 0.025_{1/m^3} . \quad (148)$$

Estimates of the uncertainties in the contributions from higher excitations and of the magnitude of the $\mathcal{O}(1/m_Q^3)$ corrections are not very firm and are rather on the optimistic side; they can be larger. With $1/m_Q^2$ corrections amounting to $\sim 8\%$ one expects $|\delta_{1/m^3_Q}| \gtrsim 0.025$ simply on the dimensional grounds.

5.1.3 Quantum-mechanical interpretation

We already illustrated the quantum-mechanical meaning of the inequality between μ_π^2 and μ_G^2 in Sect. 3.2. Let us look at the sum rule Eq. (144) from the similar perspective [4]. From the point of view of light cloud in B meson the semileptonic decay of the b quark is an instantaneous replacement of b by c quark. In ordinary quantum mechanics the overall probability of the produced state to hadronize to some final state is exactly unity, which is the first, leading term in the r.h.s. of Eq. (144). Why then are there any nonperturbative corrections in the sum rule? The answer is that the ‘normalization’ of the weak current $\bar{c}\gamma_\mu\gamma_5 b$ is not exactly unity and depends, in particular, on the external gluon field. This appears as presence of local higher-dimension operators in the current. Indeed, expressing the QCD current in terms of the nonrelativistic fields used in QM one has

$$\bar{c}\gamma_k\gamma_5 b \leftrightarrow \varphi_c^+ \left\{ \sigma_k - \left(\frac{(\vec{\sigma}i\vec{D})^2 \sigma_k}{8m_c^2} + \frac{\sigma_k (\vec{\sigma}i\vec{D})^2}{8m_c^2} - \frac{(\vec{\sigma}i\vec{D}) \sigma_k (\vec{\sigma}i\vec{D})}{4m_c m_b} \right) + \mathcal{O}\left(\frac{1}{m^3}\right) \right\} \varphi_b . \quad (149)$$

The weak current $\bar{c}\gamma_5\gamma_k b$, according to Eq. (149) converts the initial wavefunction Ψ_b into $\tilde{\Psi}$:

$$\Psi_B \xrightarrow{\bar{c}\gamma_5\gamma_k b} \tilde{\Psi} = \sigma_k \Psi_B - \left(\frac{(\vec{\sigma}i\vec{D})^2 \sigma_k}{8m_c^2} + \frac{\sigma_k (\vec{\sigma}i\vec{D})^2}{8m_c^2} - \frac{(\vec{\sigma}i\vec{D})\sigma_k(\vec{\sigma}i\vec{D})}{4m_c m_b} + \dots \right) \Psi_B. \quad (150)$$

Then it is easy to calculate the normalization of $\tilde{\Psi}$:

$$\|\tilde{\Psi}\|^2 = \|\Psi_B\|^2 - \frac{\mu_G^2}{3m_c^2} - \frac{\mu_\pi^2 - \mu_G^2}{4} \left(\frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_c m_b} \right) - \dots, \quad \|\Psi_B\|^2 = 1. \quad (151)$$

The additional terms are just the nonperturbative correction in the right-hand side of the sum rule. It is worth noting that the first two $1/m_Q^2$ terms in the bracket in Eq. (149) are the result of the Foldy-Wouthuysen transformation Eq. (17). Additional discussion, including the pedagogical illustration of the role of the latter transformation in the $1/m_Q$ expansion, can be found in Ref. [2].

6 OPE for Inclusive Weak Decays

Inclusive widths of the heavy flavor hadrons are examples of the genuine short-distance processes. The decays proceed at the space-time intervals $\sim 1/m_b$ (more precisely, inverse energy release), and the widths are affected by the soft strong dynamics to the minimal extent. The decay widths are, perhaps, the best studied example of applying OPE to quantify dynamical effects in heavy quarks. Heavy quark symmetry *per se* is of little help here, in particular in $b \rightarrow u$ or nonleptonic transitions.

The general consideration of inclusive weak decays runs parallel to the treatment of $\sigma(e^+e^- \rightarrow \text{hadrons})$. One describes the decay rate into an inclusive final state f in terms of the imaginary part of a forward scattering operator (the so-called transition operator) evaluated to second order in the weak interactions [39]:

$$\text{Im} \hat{T}(Q \rightarrow f \rightarrow Q) = \text{Im} \int d^4x \, i T \left\{ \mathcal{L}_w(x), \mathcal{L}_w^\dagger(0) \right\} \quad (152)$$

where T denotes the time ordered product and \mathcal{L}_w is the relevant weak Lagrangian at the normalization point higher or about m_Q . The space-time separation x in Eq. (152) is fixed by the inverse energy release. If the latter is sufficiently large in the decay, one can express the nonlocal operator product in Eq. (152) as an infinite sum of local heavy quark operators O_i of increasing dimensions. The width for $H_Q \rightarrow f$ is then obtained by averaging $\text{Im} \hat{T}$ over the heavy flavor hadron H_Q ,

$$\frac{\langle H_Q | 2 \text{Im} \hat{T}(Q \rightarrow f \rightarrow Q) | H_Q \rangle}{2M_{H_Q}} = \Gamma(H_Q \rightarrow f) = \frac{G_F^2 m_Q^5 |V_{\text{KM}}|^2}{192\pi^3} \sum_i \tilde{c}_i^{(f)}(\mu) \frac{\langle H_Q | O_i | H_Q \rangle_\mu}{2M_{H_Q}} \quad (153)$$

with V_{KM} denoting the appropriate combination of the CKM parameters. A few comments are in order to elucidate the content of Eq. (153).

(i) The parameter μ in Eq. (153) is the normalization point, indicating that we explicitly evolved from m_Q down to μ . The effects of momenta below μ are lumped into the matrix elements of the operators O_i .

(ii) The coefficients $\tilde{c}_i^{(f)}(\mu)$ are dimensionful, they contain powers of $1/m_Q$ that go up with the dimension of the operator O_i . Using the normalization introduced in Eq. (153), one obtains on dimensional grounds

$$\tilde{c}_i^{(f)}(\mu) \frac{1}{2M_{H_Q}} \langle H_Q | O_i | H_Q \rangle_{(\mu)} \sim \mathcal{O} \left(\frac{\Lambda_{\text{QCD}}^{d_i-3}}{m_Q^{d_i-3}}, \frac{\alpha_s \mu^{d_i-3}}{m_Q^{d_i-3}} \right)$$

with d_i denoting the dimension of operator O_i . The contribution from the operators with the lowest dimension obviously dominates in the limit $m_Q \rightarrow \infty$.

(iii) It seems natural then that the expansion of total rates can be given in powers of $1/m_Q$. The master formula (153) holds for a host of different integrated heavy-flavor decays: semileptonic, nonleptonic and radiative transitions, CKM-favored or suppressed, etc. For semileptonic and nonleptonic decays, treated through order $1/m_Q^3$, it takes the following form:

$$\begin{aligned} \Gamma(H_Q \rightarrow f) = & \frac{G_F^2 m_Q^5}{192\pi^3} |V_{\text{KM}}|^2 \times \\ & \left[c_3^{(f)}(\mu) \frac{\langle H_Q | \bar{Q}Q | H_Q \rangle_{(\mu)}}{2M_{H_Q}} + c_5^{(f)}(\mu) \frac{1}{m_Q^2} \frac{\langle H_Q | \bar{Q} \frac{1}{2} \sigma G Q | H_Q \rangle_{(\mu)}}{2M_{H_Q}} + \right. \\ & \left. + \sum_i c_{6,i}^{(f)}(\mu) \frac{1}{m_Q^3} \frac{\langle H_Q | (\bar{Q} \Gamma_i q)(\bar{q} \Gamma_i Q) | H_Q \rangle_{(\mu)}}{2M_{H_Q}} + \mathcal{O}(1/m_Q^4) \right], \end{aligned} \quad (154)$$

where Wilson coefficients $c^{(f)}$ are of order unity.

We pause here to make a few explanatory remarks on this particular expression. First, the main statement of the OPE is that there is no correction of order $1/m_Q$ [18]. This is particularly noteworthy because the hadron masses, which control the phase space, do contain such a correction: $M_{H_Q} = m_Q (1 + \bar{\Lambda}/m_Q + \mathcal{O}(1/m_Q^2))$; the parameter $\bar{\Lambda}$, different for different hadrons, does not enter the width. The reason for the absence of the $1/m_Q$ correction in the total widths is twofold: the corrections to the expectation value of the leading QCD operator $\bar{Q}Q_{(\mu)}$ is only $\sim \mu^2/m_Q^2$, and there is no independent QCD operator of dimension 4 for forward matrix elements. Since the coefficients functions are purely short-distance, infrared effects neither can penetrate into them.

A physically more illuminating way to think of the absence of corrections of order $1/m_Q$ is to realize that the bound-state effects in the initial state (mass shifts, etc.) do generate corrections of order $1/m_Q$ to the total width – as does hadronization in the final state. Yet local color symmetry demands that they cancel against each other, as can explicitly be demonstrated in simple models. It is worth realizing that this is a peculiar feature of QCD interactions – other dynamical realizations of strong confining forces would, generally, destroy the exact cancellation. A detailed

pedagogical discussion of physics behind this cancellation can be found in Ref. [2], Sect. 3.1.

Second, the leading nonperturbative corrections are $\sim \mathcal{O}(1/m_Q^2)$, i.e. small in the total decay rates for beauty hadrons. The first calculation of the leading non-perturbative corrections in the decays of heavy flavors was done in [18, 62, 63, 64].

Third, the four-quark operators $(\bar{Q}\Gamma q)(\bar{q}\Gamma Q)$ with different Lorentz and color structure depend explicitly on the light-quark flavors denoted by q . They, therefore, generate differences in the weak transition rates for the different hadrons of a given heavy flavor.¹³ They describe the effects of Weak Annihilation (WA) and Pauli Interference in B mesons, and Pauli Interference and Weak Scattering (WS) in heavy baryons. Their effects were calculated already in mid-eighties [47].

A note should be made regarding the subtlety in understanding the above four-quark expectation values [65, 66]. They must be taken in the effective low energy (nonrelativistic) theory with respect to the heavy quark Q . In particular, the heavy quark field $Q(x)$ must contain only the heavy quark annihilation operator, while $\bar{Q}(x)$ only the creation of the heavy quark. This differs from the full QCD fields which contain both annihilation of the heavy quark and creation of the antiquark in the $Q(x)$ field (and likewise for $\bar{Q}(x)$). The full-QCD four quark expectation values over, say, B meson $\langle B|\bar{b}\Gamma_1 q \bar{q}\Gamma_2 b|B\rangle$ include, strictly speaking, the intermediate states with the $b\bar{b}$ pair (not related to the gluon conversion into $b\bar{b}$), and not only light flavor hadrons. Even though the energy gap between the two classes of states is large, about $2m_Q$, the latter contributions to the expectation values are not necessarily short-distance and rather governed by strong coupling dynamics when their energy is close to $2m_Q$. This additional nonperturbative component is absent from the expectation values in the effective low-energy theory in the sector with a single heavy quark. Simultaneously, the analysis shows that such contributions are absent from the inclusive widths as well. Therefore, strictly speaking one cannot “match” in the usual sense the full-QCD expectation values onto those in the effective theory. The renormalization scale evolution of the four-fermion operators in the effective theory is governed by the so-called “hybrid” anomalous dimensions introduced by Shifman and Voloshin in the mid 80s [47].

Fourth, the short-distance coefficients $c_i^{(f)}(\mu)$ in practice are calculated in perturbation theory. However it is quite conceivable that certain nonperturbative effects arise also in the short-distance regime. They are believed to be rather small in beauty decays [67].

Fifth, a new matrix element appearing in OPE, not discussed so far, is the scalar heavy quark density. Its nonrelativistic expansion originally established in Ref. [18] using the QCD equations of motion for the heavy quark field, follows from the identity (16) and takes the form

$$\langle H_Q|\bar{Q}Q|H_Q\rangle = \langle H_Q|\bar{Q}\gamma_0 Q|H_Q\rangle - \frac{\langle H_Q|\bar{Q}\left(\vec{\pi}^2 - \frac{i}{2}\sigma G\right)Q|H_Q\rangle}{2m_Q^2} + \mathcal{O}(1/m_Q^4) . \quad (155)$$

¹³Expanding $\langle H_Q|\bar{Q}i\sigma G Q|H_Q\rangle/m_Q^2$ also yields contributions of order $1/m_Q^3$; those are, however, practically insensitive to the light quark flavors.

Since $\langle H_Q | \bar{Q} \gamma_0 Q | H_Q \rangle = 2M_{H_Q}$, the spectator ansatz indeed emerges as the asymptotic scenario universal for all types of hadrons, and holds up to $1/m_Q^2$ corrections. In addition to $\bar{Q} \pi^2 Q$, the second dimension-five operator is the chromomagnetic operator $\bar{Q} i \sigma G Q$. Since $\bar{Q} \vec{D}^2 Q$ is not a Lorentz scalar, it does not appear independently in Eq. (154). For the pseudoscalar mesons, for example, we have

$$\frac{1}{2M_{P_Q}} \langle P_Q | \bar{Q} Q | P_Q \rangle = 1 - \frac{\mu_\pi^2}{2m_Q^2} + \frac{3}{8} \frac{M_{V_Q}^2 - M_{P_Q}^2}{m_Q^2} + \mathcal{O}(1/m_Q^3) . \quad (156)$$

The reason for the kinetic operator term to appear is quite transparent. The first two quantities on the right-hand side of the equation represent the mean value of the factor $\sqrt{1-\vec{v}^2}$ reflecting the time dilation which slows down the decay of the quark Q moving inside H_Q [64, 65, 68].

Equation (155) is readily obtained in the heavy quark expansion if one uses proper nonrelativistic heavy quark spinors incorporating the Foldy-Wouthuysen transformation; in the context of HQET this procedure was advocated in Ref. [69], but was largely ignored for a few years.

Finally, Eqs. (154)–(155) show that the two dimension-five operators do produce differences in B versus Λ_b/Ξ_b versus Ω_b decays of order $1/m_Q^2$. To a small extent they can also differentiate B and B_s *via* the $SU(3)$ breaking in their expectation values. Differences in the transition rates inside the meson family are generated at order $1/m_Q^3$ by dimension-six four-quark operators. They are usually estimated in the vacuum saturation approximation which – although cannot be exact – represents a reasonable starting approximation. There is an intriguing way to check factorization experimentally [65]: similar four-fermion operators enter semileptonic $b \rightarrow u$ transition rates. Moreover, in the heavy quark limit the four-fermion operators populate mainly the transitions into the hadronic states with low energy, and thus show up, for example, in the end-point domain of the lepton spectrum where their relative effect is enhanced. More precisely, these effects are present at large values of lepton invariant mass q^2 and thus can be isolated in the cleanest way studying double differential semileptonic distributions. Considering the difference of the decay characteristics of the charged and neutral B ’s in this domain, one can measure these matrix elements and even feel their scale dependence. Further details regarding the “flavor-dependent” preasymptotic effects can be found in dedicated papers [70, 71].

It is important to keep in mind that the OPE approach discussed in this section implies that *all* decay channels induced by a given term in the short distance Lagrangian in Eq. (152) are included. It is not enough to consider the states that appear at the free quark level. The final state interactions can annihilate, for example, the $\bar{c}c$ quark pair into light hadrons; electromagnetic interaction, if considered, can do the same. Disregarding the channels that can emerge due to such final state interactions can violate the general theorems: even $1/m_Q$ terms can appear in such incomplete “inclusive” widths. For example, the correction to the $b \rightarrow s + \gamma$ width does have nonperturbative corrections scaling like $1/m_b$ due to the effect of weak annihilation (in mesons, or weak scattering in baryons) of the light quarks with

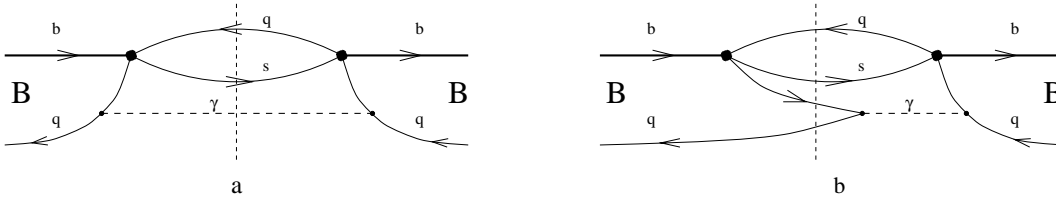


Figure 8: **a)** Spectator effect leading to $1/m_b$ correction in the decay width $b \rightarrow s + \gamma$. **b)** A different cut of the same diagram leads to an electromagnetic correction in the hadronic decay width. Terms $1/m_b$ and $1/m_b^2$ cancel out in the sum of two decay widths. The solid dot in the vertex denotes the penguin-induced $b \rightarrow s\bar{q}q$ interaction.

emission of a hard photon in the penguin-induced weak decay $b \rightarrow s\bar{q}q$, see Fig. 8a. This effect would only cancel against the (virtual) electromagnetic correction to the hadronic penguin-induced width, Fig. 8b. [72, 65].

For the very same reason the spectator-independent nonperturbative effects in the actual $b \rightarrow s + \gamma$ width can also have in practice small corrections which scale like $1/m_b$ [73]. The effective $b \rightarrow s + \gamma$ interaction is generated by loop diagrams and, including contributions from $c\bar{c}$ ($u\bar{u}$) pairs with momenta not large compared to m_b , is not genuinely pointlike. This part of the decay interaction is properly treated considering the original $b \rightarrow c\bar{c}s$ ($b \rightarrow u\bar{u}s$) operators as the weak decay Lagrangian \mathcal{L}_w , and then truly inclusive widths for such an interaction have to include the probability of the usual decay $b \rightarrow c\bar{c}s$. The latter is affected by electromagnetic interaction, again by virtue of the same diagram as for the $b \rightarrow s + \gamma$ decay, but with a different cut. Of course, in experiment – in contrast to the OPE – these processes are completely different and are taken separately.

The OPE predictions for the nonperturbative effects in the inclusive width differences are rather nontrivial. Consequently, a number of questions are often raised. First, is this analysis operating in terms of quarks and gluons, applicable to actual b hadrons, with confinement leading to the drastic change of the physical spectrum which consists only of the colorless hadrons but not quarks? Second, if this is correct, how to formally justify this OPE? Third, how to compute the leading nonperturbative corrections to the decay rates? And, finally, it is advantageous to have a transparent physical interpretation of the OPE machinery.

The answer to the first question is definitely positive, although is not trivial; the validity of the OPE approach has been challenged more than once, even over the recent years. The raised criticism covered different aspects, but basically reduced to one main point: due to ‘brutal confinement’ there is no “duality” between the OPE computations operating in terms of quarks and gluons, and actual hadronic decay rates, either for nonleptonic, or for any type of the underlying quark transitions. We do not dwell here on discussing the reasons (usually poorly substantiated) behind such suggestions. This question was recently discussed in much detail in the context

of exactly solvable two-dimensional 't Hooft model where, in principle, all hadron masses and decay amplitudes can be computed. In Refs. [74, 66, 75] the analytic summation of the rates for open decay channels was performed in the expansion in powers of $1/m_Q$, and compared to the OPE predictions for the model. It was shown that to all orders in $1/m_Q$ the OPE series could be computed, they reproduced exactly the asymptotic expansion of the actual inclusive widths, for both semileptonic and nonleptonic decays.

The physical interpretation of the main OPE result, the absence of the $1/m_Q$ corrections to the decay widths has been briefly mentioned above. A dedicated discussion can be found in Ref. [2], Sect. 3.1. In the following sections we briefly address the third and the second questions, respectively.

6.1 Sample computation

Here we illustrate the computation of the leading nonperturbative corrections in semileptonic, nonleptonic and $b \rightarrow s + \gamma$ -type decays. For simplicity, we do not go beyond order $1/m_b^2$, and also neglect masses of the final state quarks and leptons; this will make expressions more compact. For semileptonic and nonleptonic decays we still will refer to one of the final state quarks as the charmed one. The corresponding decay Lagrangian is given (we omit the CKM factors)

$$\mathcal{L}_w = -\frac{G_F}{\sqrt{2}} \bar{c} \gamma_\mu (1 - \gamma_5) b \cdot \bar{d} \gamma_\mu (1 - \gamma_5) u ; \quad (157)$$

this will describe the semileptonic decays as well if we switch off strong interaction of the $d\bar{u}$ pair. We do not introduce here the color contraction schemes explicitly; color indices can be easily taken care of in the end. For $b \rightarrow s + \gamma$ we take the decay Lagrangian in the form

$$\mathcal{L}_w = \frac{\lambda}{2} \bar{s} i \sigma_{\mu\nu} F^{\mu\nu} (1 + \gamma_5) b . \quad (158)$$

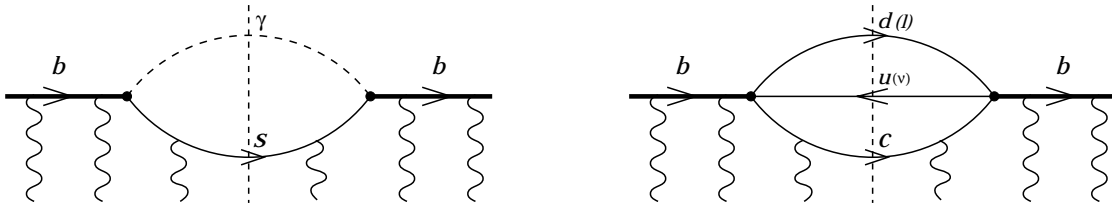


Figure 9: Diagrams describing the the transition amplitude in the external field for computing the $b \rightarrow s + \gamma$ (left) and nonleptonic or semileptonic widths (right). Interaction with the gluon medium enters through Green functions for the final state quarks (intermediate quark lines), and through equations of motion for the initial b fields.

The computation of the transition operator \hat{T} in Eq. (152) is most simply done in the Fock-Schwinger (fixed point) gauge $x_\mu A_\mu(x) = 0$ (see Ref. [76] for details).

The transition operator in the coordinate space is given by the products of the fermion and photon Green functions like $\bar{b}(x) G(x, 0) \cdot D(x, 0) b(0)$, or by the product of three fermion Green functions $\bar{b}(x) G(x, 0) \cdot G(x, 0) \cdot G(0, x) b(0)$ for the two cases, respectively. For massless particles the Green functions are

$$G(x, 0) = \frac{i\not{x}}{2\pi^2 x^4} - \frac{ix_\alpha \tilde{G}_{\alpha\beta} \gamma_\beta \gamma_5}{8\pi^2 x^2} + \dots, \quad D_{\mu\nu}(x, 0) = \frac{\delta_{\mu\nu}}{4\pi^2 x^2} \quad (159)$$

(a note of caution: in the fixed point gauge one should distinguish $G(x, 0)$ from $G(0, x)$ for quark Green functions), and the expansion of the gauge field is given by

$$A_\mu(x) = \frac{1}{2 \cdot 0!} x_\nu G_{\nu\mu}(0) + \frac{1}{3 \cdot 1!} x_\nu x_\rho (D_\rho G_{\nu\mu}) + \dots \quad (160)$$

\tilde{G} above is the dual field strength, $\tilde{G}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta}$. Note that we work directly in Minkowski space. For $b \rightarrow s + \gamma$ we thus have

$$\begin{aligned} \hat{T}_{bs\gamma} = 4i\lambda^2 \frac{x^2 \delta_{\mu\rho} - 4x_\mu x_\rho}{2\pi^2 x^6} \bar{b}(x) \sigma_{\mu\nu} \left[\frac{i\not{x}}{2\pi^2 x^4} - \frac{ix_\alpha \tilde{G}_{\alpha\beta} \gamma_\beta \gamma_5}{8\pi^2 x^2} \right] \sigma_{\rho\nu} (1 + \gamma_5) b(0) = \\ -\lambda^2 \bar{b}(x) \left[\frac{6}{\pi^4} \frac{\not{x}}{x^8} + \frac{1}{2\pi^4} \frac{x_\alpha}{x^6} \tilde{G}_{\alpha\beta} \gamma_\beta + \dots \right] (1 + \gamma_5) b(0), \end{aligned} \quad (161)$$

where the summation over μ, ν is straightforward. For the decays mediated by the four-fermion interaction we consider both semileptonic and nonleptonic cases simultaneously, simply introducing the factor ξ which would switch off interaction of leptons with the gluon field, $\xi=1$ for $b \rightarrow c \bar{u} d$ and $\xi=0$ for $b \rightarrow c \ell \bar{\nu}$:

$$\begin{aligned} \hat{T}_{4\text{-ferm}} = 4iG_F^2 \bar{b}(x) \gamma_\mu \left(\frac{i\not{x}}{2\pi^2 x^4} - \frac{ix_\alpha \tilde{G}_{\alpha\beta} \gamma_\beta \gamma_5}{8\pi^2 x^2} \right) \gamma_\nu \left(-\frac{i\not{x}}{2\pi^2 x^4} - \xi \frac{ix\tilde{G}\gamma\gamma_5}{8\pi^2 x^2} \right) \cdot \\ \cdot \gamma_\mu \left(\frac{i\not{x}}{2\pi^2 x^4} - \xi \frac{ix\tilde{G}\gamma\gamma_5}{8\pi^2 x^2} \right) \gamma_\nu (1 - \gamma_5) b(0) = \\ 4G_F^2 \bar{b}(x) \left[\frac{\not{x}}{\pi^6 x^{10}} + \xi \frac{x_\alpha}{4\pi^6 x^8} \tilde{G}_{\alpha\beta} \gamma_\beta + \dots \right] (1 - \gamma_5) b(0), \end{aligned} \quad (162)$$

where we have used the Fiertz transformation for one of the $V-A$ vertices, and the short-hand notation $x\tilde{G}\gamma$ denotes the repeating structure $x_\alpha \tilde{G}_{\alpha\beta} \gamma_\beta$. We observe that only the interaction with the antiquark is present to this order (using a variant of the Fiertz transformation, it is easy to show that this holds for arbitrary fermion masses [18]).

Now we need to make the Fourier transformation to pass to the momentum representation of the transition operator:

$$\int d^4x e^{-ipx} \frac{x_\mu}{x^6} = -\frac{\pi^2}{4} p_\mu \ln(-p^2) + \text{polynomial in } p^2 \quad (163)$$

$$\int d^4x e^{-ipx} \frac{x_\mu}{x^8} = \frac{\pi^2}{48} p^2 p_\mu \ln(-p^2) + \text{polynomial in } p^2 \quad (164)$$

$$\int d^4x e^{-ipx} \frac{x_\mu}{x^{10}} = -\frac{\pi^2}{64 \cdot 24} p^4 p_\mu \ln(-p^2) + \text{polynomial in } p^2 \quad (165)$$

(the divergent pieces come from infinite momenta running inside loops and cannot yield imaginary part). The polynomials do not have discontinuity and can be discarded for computing decay widths; it comes only from the logarithm.

Now we need to evaluate the expectation values $\bar{b} p^4 \not{p} b$, $\bar{b} p^2 \not{p} b$ etc. which enter at $p^2 \simeq m_b^2$. To do this we recall that $p_\mu = iD_\mu - A_\mu$ and in the Fock-Schwinger gauge one has

$$A_\mu(0) = 0, \quad D_\mu A_\nu|_0 = \frac{1}{2} G_{\mu\nu}(0) \quad (166)$$

(this implies, for example, that in our approximation $[D_\mu, A_\mu]$ can be replaced by zero). Our strategy is to pull the gluon potential A_μ to the left since $A(0) = 0$ and the result does not vanish only if a derivative acts on A_μ . In this way we arrive at the very simple rule that in this gauge

$$\bar{b} p^{2n} \not{p} b = m_b^{2n+1} \bar{b} b - n m_b^{2n-1} \bar{b} \frac{i}{2} \sigma G b + \mathcal{O}(m_b^{2n+1}/m_b^3). \quad (167)$$

We also have an additional structure which, for the forward matrix elements reduces to the chromomagnetic operator using equations of motion for the b field:

$$\bar{b} p_\alpha \tilde{G}^{\alpha\beta} \gamma_\beta \gamma_5 b = m_b \bar{b} \frac{i}{2} \sigma G b.$$

Recalling that $\Gamma = 2 \text{Im} T$ and collecting all terms, we arrive at [18, 63]

$$\begin{aligned} \Gamma_{\text{sl}} &= \frac{G_F^2 m_b^5}{192\pi^3} \frac{\langle B | \bar{b} b | B \rangle}{2M_B} \left\{ 1 - 2 \frac{\mu_G^2}{m_b^2} + \mathcal{O}\left(\frac{1}{m_b^3}\right) \right\}, \\ \Gamma_{b \rightarrow s + \gamma} &= \frac{\lambda^2 m_b^3}{4\pi} \frac{\langle B | \bar{b} b | B \rangle}{2M_B} \left\{ 1 - (1+1) \frac{\mu_G^2}{m_b^2} + \mathcal{O}\left(\frac{1}{m_b^3}\right) \right\}, \\ \Gamma_{\text{nl}} &= \frac{G_F^2 m_b^5 N_c}{192\pi^3} \frac{\langle B | \bar{b} b | B \rangle}{2M_B} \left\{ \left(\frac{c_+^2 + c_-^2}{2} + \frac{c_+^2 - c_-^2}{2N_c} \right) \left(1 - \frac{2\mu_G^2}{m_b^2} \right) - \frac{c_+^2 - c_-^2}{2N_c} \frac{8\mu_G^2}{m_b^2} + \mathcal{O}\left(\frac{1}{m_b^3}\right) \right\}. \end{aligned} \quad (168)$$

Account for the charm quark mass is straightforward and yields the factor $(1 - m_c^2/m_b^2)^3$ for the direct contribution of the operator μ_G^2 , the second term in Γ_{nl} . An additional piece expressed in terms of the derivative of the free quark phase space factor z_0 with respect to m_c^2 comes from translating the result from the Fock-Schwinger gauge, cf. Eq. (167); it introduces the factor $(1 - m_c^2/m_b^2)^4$ for μ_G^2 in the first term, and likewise in Γ_{sl} . The correction to the nonleptonic width $b \rightarrow c\bar{c}s$ with two heavy quarks in the final state was computed in Ref. [62].

6.2 How OPE can be justified for inclusive widths

In the previous section we illustrated the calculation of the leading nonperturbative corrections to the inclusive decay widths. The expected question is: what such a computation of the quark decay width in the external gluon field has to do with the decay width of the actual heavy flavor hadron? The justification does not differ

conceptually from, say, the classical case of e^+e^- annihilation to hadrons and makes use of the analytic properties of the transition amplitude and the dispersion relations. This was discussed in Refs. [65, 67, 74, 66] with varying degree of detalization.

To study the analytic properties of the forward transition amplitude we must introduce an auxiliary complex variable ω :

$$\mathcal{A}(\omega) = \int d^4x \, e^{-i\omega(vx)} \langle H_Q | iT \{ \mathcal{L}_w(x), \mathcal{L}_w^\dagger(0) \} | H_Q \rangle \quad (169)$$

(once again, v is the four-velocity of the decaying heavy hadron). This ω -dependent amplitude can be visualized as the transition amplitude governing the total (weak) cross section of the scattering of a fictitious spurion particle S on the heavy quark,

$$S(q) + H_Q(p) \rightarrow \text{light hadrons} \, , \quad (170)$$

or the weak decay width in the process

$$Q \rightarrow \text{quarks (leptons)} + S \, . \quad (171)$$

Such processes would appear if the weak decay Lagrangian is modified from, say the conventional four-fermion form to the “four-fermion + spurion” interaction,

$$\mathcal{L}_w(x) \rightarrow S(x) \mathcal{L}_w(x) \, . \quad (172)$$

For simplicity it is convenient to assume, as in Eq. (169) that the spurion field does not carry spacelike momentum but only energy.

The amplitude $\mathcal{A}(\omega)$ has the usual analytic properties: it is analytic and has a number of cuts describing the physical processes in different channels (s , t or u). The physical cut corresponding to the weak decay of the heavy quark we are interested in, starts near $\omega \simeq E_r$ where the energy release E_r denotes m_Q minus the sum of the masses of the final state quarks and/or leptons. Other cuts are located far enough from this point and from the physical point $\omega = 0$. The discontinuity across the physical cut at which the point $\omega = 0$ is located, describes the total decay width we are interested in. The OPE for the inclusive widths relies on the fact that the short-distance expansion of $\mathcal{A}(\omega)$ runs in $1/(\omega - E_r)$ and can be applied near the physical point $\omega = 0$ exactly as in e^+e^- annihilation near a positive value of $s \gg \Lambda_{\text{QCD}}^2$. To the same extent, in principle, a certain smearing can be required if the hadronic probabilities still exhibit the resonance structure.

Thus, there is no theoretical peculiarity in the asymptotic applications of the OPE, say for nonleptonic widths compared to semileptonic. It does not make a conceptual difference to perform a short-distance expansion of a single quark Green function (like in semileptonic widths or $b \rightarrow s + \gamma$), of the product of two Green functions (e^+e^- annihilation) or of the product of three quark Green functions (the nonleptonic widths).

Smearing in ω can often be phrased as smearing over the interval of m_Q . Indeed, in the heavy quark limit the decay amplitudes depend on just the combination $m_b - \omega$, therefore

$$\mathcal{A}(\omega, m_Q) \simeq \mathcal{A}(0, m_Q - \omega) \quad (173)$$

(there are power corrections to this relation associated with explicit mass effects in the initial state). Smearing in the heavy quark mass may look more transparent physically when the effect of opening new threshold is effectively averaged.

The structure of the cuts in the amplitude $\mathcal{A}(\omega)$ is particularly simple if all the masses of the final state quarks are large compared to Λ_{QCD} . What happens if some final state quarks are light, for example, even in semileptonic $b \rightarrow u$ decays? The physical cut corresponding to the decay into u quark can come very close to the cut which would describe the scattering of \bar{u} on the heavy quark – the distance between them is twice the energy of the quark, and it can be small if the heavy quark line is soft (i.e., both energy and spacelike momentum are small). We would not be able to distinguish between the contributions of the two processes if their respective cuts are too close in the scale of strong interactions. The answer is that in the OPE we remove explicitly all soft lines (including the quark ones) which carry small momenta; their contributions are described by the “condensates” where the soft legs are treated as the external fields acting on the initial state. The remaining pieces describe only energetic particles for which the proximity of the cuts cannot take place. From this perspective the OPE cutoff μ in computing the Wilson coefficients acts similar to assigning a mass μ to each potentially soft quark line.

It is important to emphasize in this respect that the OPE treatment of these contributions in the “corners of the phase space” differs from, say, the computation of the regular $1/m_Q^2$ terms given by the chromomagnetic operator. In the latter case the effect (at least a part of it) comes from the kinematics where the corresponding final state quark is hard, and we really use the short-distance expansion of the propagator to compute this term. In the case of soft quark legs we do not compute them in the OPE, but, in a sense, parameterize their contribution. The large momentum flowing through the rest of the diagram only ensures that it is given by the expectation value of a *local* heavy quark operator. However, even if we knew the full transition amplitude with some accuracy near $\omega = 0$, we still would not be able to resolve the contribution of the physical and the “ u -channel” cuts. These subtleties used to cause some confusion in the literature; the proximity of the cuts was viewed as the failure of the OPE to describe such contributions [55].

6.3 $|V_{cb}|$ from the total semileptonic B width

Including the leading corrections, the semileptonic width has the following form [18, 62, 63, 64]:

$$\Gamma_{\text{sl}} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left\{ z_0 \left(1 - \frac{\mu_\pi^2 - \mu_G^2}{2m_b^2} \right) - 2 \left(1 - \frac{m_c^2}{m_b^2} \right)^4 \frac{\mu_G^2}{m_b^2} - \frac{2}{3} \frac{\alpha_s}{\pi} a_1 + \dots \right\} \quad (174)$$

where ellipses stand for higher order perturbative and/or power corrections, z_0 and a_1 depend on m_c^2/m_b^2 . Regardless the exact value of μ_π^2 , the direct $1/m_b^2$ corrections to Γ_{sl} are rather small, about -5% and lead to the increase in the extracted value of $|V_{cb}|$ by 2.5% . The higher order power corrections are at a percent level. There

is one new operator, the Darwin term which appears to order $1/(m_b - m_c)^3$; it was evaluated in Refs. [77, 78] and may decrease the width by up to 4%. The remaining $1/m_b^3$ corrections enter only through the $1/m_b$ terms in the expectation values of the kinetic and chromomagnetic operators in actual B mesons. They include the ‘spin-orbital’ expectation value ρ_{LS}^3 as a $1/m_b$ part of the full QCD operator $\bar{b}_2^i \sigma G b$, and the nonlocal correlators introduced in Sect. 2.2. Their effect is minor and is included in the uncertainties of the chromomagnetic and kinetic expectation values.

The leading-order $\mathcal{O}(\alpha_s)$ perturbative corrections are known from the QED calculations in muon decay [79]. The BLM corrections [80], a part of the higher-order perturbative series associated only with running of α_s in the first-order loop diagrams, were calculated to all orders [81]. Their impact appeared to be small if the width is expressed in terms of the properly defined short-distance masses [82]. The remaining non-BLM (‘genuine’) two-loop α_s^2 corrections were expected to be moderate if one uses the heavy quark masses normalized at the scale around 1 GeV [22]. They were evaluated in a series of papers by Czarnecki and Melnikov [59, 83] and were confirmed to be small. At present, no significant theoretical uncertainty remains in the perturbative corrections to the semileptonic width.

The running mass of the b quark is known with high precision, Sect. 3.1.2. To determine m_c , we can rely on relation (35) for the difference between m_b and m_c in terms of the hadron masses. It turns out that in this way the direct dependence of the semileptonic width on ρ_D^3 almost cancels against the one coming indirectly from $m_b - m_c$, therefore the width becomes sensitive only to the exact values of μ_π^2 and $\bar{\rho}^3$.

Evaluating the theoretical prediction we get

$$|V_{cb}| = 0.0412 \left(\frac{\text{BR}(B \rightarrow X_c \ell \nu)}{0.105} \right)^{\frac{1}{2}} \left(\frac{1.55 \text{ ps}}{\tau_B} \right)^{\frac{1}{2}} \cdot \left(1 - 0.012 \frac{(\mu_\pi^2 - 0.5 \text{ GeV}^2)}{0.1 \text{ GeV}^2} \right) \times \\ \left(1 - 0.01 \frac{\delta m_b(1 \text{ GeV})}{50 \text{ MeV}} \right) \left(1 + 0.007 \frac{\bar{\rho}^3}{0.1 \text{ GeV}^3} \right). \quad (175)$$

The main theoretical uncertainty at the moment resides in the value of μ_π^2 , which comes from constraining $m_b - m_c$ in the heavy quark expansion. It is worth noting that this is the only place where we relied on the expansion in $1/m_c$. It is vulnerable to possible late onset of the $1/m_Q$ expansion which would show up as the large expectation values of higher-dimension operators. Yet we note that the meson masses are expected to be the most robust observables; it is well known in ordinary quantum mechanics that eigenvalues of the Hamiltonian are more stable than wavefunctions and their overlaps. Nevertheless, it seems important to have an independent direct determination of m_c to isolate this potential problem.

Finally, we arrive at the model-independent evaluation

$$|V_{cb}| = 0.0412 \left(\frac{\text{BR}(B \rightarrow X_c \ell \nu)}{0.105} \right)^{\frac{1}{2}} \left(\frac{1.55 \text{ ps}}{\tau_B} \right)^{\frac{1}{2}} \times$$

$$\left(1 - 0.012 \frac{(\mu_\pi^2 - 0.5 \text{ GeV}^2)}{0.1 \text{ GeV}^2}\right) \cdot (1 \pm 0.012_{\text{pert}} \pm 0.01_{m_b} \pm 0.02) , \quad (176)$$

where the last error reflects m_Q^{-3} and higher power corrections as well as possible deviations from local duality.

6.4 $\Gamma_{\text{sl}}(b \rightarrow u)$ and determination of $|V_{ub}|$

Similar to the treatment of $\Gamma(B \rightarrow X_c \ell \nu)$, it is straightforward to relate the value of $|V_{ub}|$ to the total semileptonic width $\Gamma(B \rightarrow X_u \ell \nu)$. The dedicated analysis was performed in Ref. [84]:

$$|V_{ub}| = 0.00445 \left(\frac{\text{BR}(B^0 \rightarrow X_u \ell \nu)}{0.002} \right)^{\frac{1}{2}} \left(\frac{1.55 \text{ ps}}{\tau_B} \right)^{\frac{1}{2}} \cdot (1 \pm 0.01_{\text{pert}} \pm 0.03_{m_b} \pm 0.015_{\text{nonpert}}) . \quad (177)$$

The dependence on μ_π^2 is practically absent here. The complete perturbative corrections are known analytically two two orders [85], and again turn out to be small if one uses $m_b(1 \text{ GeV})$ as the input. The contribution at the level of a few percent can be expected from nonfactorizable expectation values of the four-quark operators at order $1/m_b^3$. At this level it would be advantageous to measure the $b \rightarrow u$ semileptonic decay width for the charged and neutral B mesons separately [65].

The most direct way to disentangle $b \rightarrow u \ell \nu$ decays from hundred times more abundant $b \rightarrow c \ell \nu$ without tagging the secondary charm decay would be to study the invariant mass of hadrons in the final state, M_X :

$$\frac{d}{dM_X} \Gamma(B \rightarrow X \ell \nu) , \quad M_X^2 = \left(\sum_i P_{\text{hadr}}^{(i)} \right)^2 . \quad (178)$$

For the free quark decay one has $M_X^2 \simeq 0$ in $b \rightarrow u$ and $M_X^2 = m_c^2$ for the $b \rightarrow c$ transitions. In actual decays the mass can take values exceeding m_π and M_D , respectively. The increase in mass can originate both perturbatively if a hard gluon is emitted in the decay, or through soft bound-state or hadronization processes.

The leading soft effects turn out to significantly modify the M_X distribution due to the effects of primordial ‘motion’ of the heavy quark caused by bound state dynamics in the initial b hadron. It was pointed out already in [64] that the QCD-based OPE automatically leads to an analogue of this “Fermi motion” which had been introduced phenomenologically long ago, first in [86] and then elaborated further to the status of a well-formulated model in [87]. A detailed description of the Fermi motion in the framework of the $1/m_Q$ expansion was later given in [30, 88] and in [89]; it goes outside the scope of the present review. Here we only briefly mention the basic facts.

The “Fermi motion” in QCD has certain peculiarities which are absent in the phenomenological models [30, 90]. The distribution over the ‘primordial’ Fermi momentum $F(\vec{p})$ is replaced by a certain distribution function $F(x)$, where $x \leq 1$ measures the momentum of the b quark in the units of $M_B - m_b$. It is similar to the

leading-twist distribution function of deep inelastic scattering (DIS) on usual light hadrons. First distinction is that $F(x)$ is one-dimensional; one can define only the distribution over a certain projection of the momentum.

Second, $F(x)$ depends essentially on the final state quark mass (more exactly, on its velocity). While it is the same for $b \rightarrow u \ell \nu$ and $b \rightarrow s + \gamma$ where the final quark is ultrarelativistic and is given by the light-cone distribution function like in DIS, it is rather different for c quark in $b \rightarrow c \ell \nu$ where it is closer to a nonrelativistic particle; in this case the light-cone distribution is replaced by the so-called temporal distribution function.

Finally, the QCD $F(x)$ is normalization-point dependent. This property is well known already from usual DIS; the evolution of the heavy quark distribution function, where normalization point is well below m_Q is, however, much stronger.

The leading soft effects generating the M_X spectrum emerge due to same physics which gives rise to the Fermi motion. They are quite significant. For example, the average invariant mass square of hadrons in $b \rightarrow u$ gets the nonperturbative correction $\sim \Lambda_{\text{QCD}} \cdot m_b$ [65]:

$$\langle M_X^2 \rangle_{\text{nonpert}} = \frac{7}{10} m_b (M_B - m_b) + \mathcal{O}(\Lambda_{\text{QCD}}^2) \simeq 1.5 \text{ GeV}^2. \quad (179)$$

The perturbative corrections lead to $\langle M_X^2 \rangle_{\text{pert}} \sim \frac{\alpha_s}{\pi} m_b^2$, however in B decays this increase is still smaller than through the nonperturbative effects.

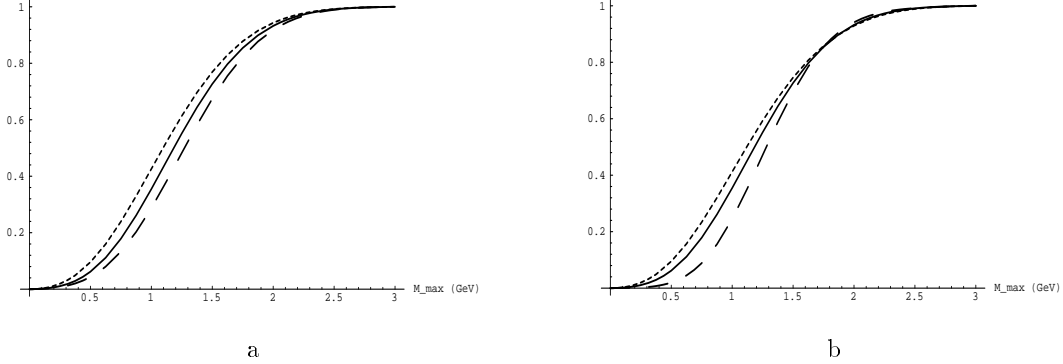


Figure 10: The integrated fraction of the $b \rightarrow u \ell \nu$ events $\Phi(M_X)$. **(a)**: Dependence on m_b which is varied within ± 50 MeV. **(b)**: Effect of variation of μ_π^2 by $\pm 0.2 \text{ GeV}^2$.

To quantify the QCD effects on the M_X distribution let us introduce, following Refs. [91, 92] the fraction of $b \rightarrow u$ events with M_X below a certain cutoff mass M_{max} :

$$\Phi(M_{\text{max}}) = \frac{1}{\Gamma(b \rightarrow u)} \int_0^{M_{\text{max}}} dM_X \frac{d\Gamma}{dM_X}. \quad (180)$$

$\Phi(0) = 0$ and $\Phi(M_B) = 1$ hold regardless of any dynamics. The main question to theory is whether it can calculate accurately enough $\Phi(M_{\text{max}})$ with $M_{\text{max}} \lesssim 1.6 \text{ GeV}$.

The dedicated analysis was carried out in [91, 92], and the conclusion appeared to be quite optimistic: $b \rightarrow u \ell \nu$ decays are not expected to populate the domain above $M_X = 1.6 \text{ GeV}$ too significantly. The typical theoretical predictions for $\Phi(M_{\text{max}})$ are shown in Fig. 10.

The corresponding experimental analysis was first attempted by ALEPH [93] and recently performed by DELPHI [94]. The M_X spectrum was found in a nice agreement with the above theoretical predictions, and the value of $|V_{ub}/V_{cb}|$ was found about 0.10, with the uncertainty of approximately 20 percentage points; the latter is still dominated by the experimental error bars.

6.5 Summary on $|V_{cb}|$

Here we give a brief summary of the two methods of extracting $|V_{cb}|$. The most precise at the moment is the value obtained from $\Gamma_{\text{sl}}(B)$: Eqs. (175) and (176) with the central theoretical input values shown there lead to

$$|V_{cb}| \simeq 0.0413 \cdot \left(1 - 0.012 \frac{(\mu_\pi^2 - 0.5 \text{ GeV}^2)}{0.1 \text{ GeV}^2} \right). \quad (181)$$

The overall relative theoretical uncertainty in this result is $\delta_{\text{th}} \lesssim 5\%$. With future refinements we can expect reducing the uncertainty down to 2%.

The $B \rightarrow D^* \ell \nu$ zero-recoil rate also provides a good accuracy. The exact experimental status of the measurements extrapolated to $\vec{q} = 0$ is not completely clear at the moment. Until summer 2000 the results from the LEP experiments and from CLEO seemed to be in a good agreement, yielding $F_{D^*}|V_{cb}| \simeq 0.035$ with a typical uncertainty of $\pm 0.001_{\text{stat}} \pm 0.002_{\text{syst}}$. Using this reported average value and the literal estimate $F_{D^*} \simeq 0.89$ only a bit lower value of $|V_{cb}| \simeq 0.0395$ emerges. Adding only one (systematic) standard deviation to $F_{D^*}|V_{cb}|$ would yield just the central value for $|V_{cb}|$ extracted from the total semileptonic width – even without varying the numbers within theoretical uncertainties.

It is interesting that the central theoretical value in this method, according to Eq. (148) exhibits the dependence on μ_π^2 similar in magnitude but opposite in sign to Eq. (181):

$$|V_{cb}| \simeq 0.0395 \cdot \left(1 + 0.015 \frac{(\mu_\pi^2 - 0.5 \text{ GeV}^2)}{0.1 \text{ GeV}^2} \right). \quad (182)$$

The theoretical uncertainty δ_{th} here constitutes probably $\delta_{\text{th}} \approx 6\%$, however this estimate relies on additional theoretical assumptions. It is not clear how it can be decreased.

Recently CLEO came up with the new value for $F_{D^*}|V_{cb}| = 0.0424 \pm 0.0018_{\text{stat}} \pm 0.0019_{\text{syst}}$ [95] which is significantly higher than both their old number and the current LEP average value $0.0345 \pm .0007_{\text{stat}} \pm 0.0015_{\text{syst}}$ [96], and now has larger error bars. Taking it literally, one even observes ‘overshooting’ – the central value for $|V_{cb}|$ under the same theoretical assumptions would be even higher than the result from $\Gamma_{\text{sl}}(B)$. Since the shift exceeds the previously quoted error bars, it may be premature to draw a definite conclusion at the moment.

In any case, it is remarkable that the values of $|V_{cb}|$ that emerged from exploiting two theoretically complementary approaches are very close. The progress was not for free: it became possible only due to essential refinements of the theoretical tools in the last several years, which prompted us, in particular, that the zero-recoil $B \rightarrow D^*$ formfactor F_{D^*} is probably close to 0.9, significantly lower than previous expectations. The decrease in F_{D^*} and more accurate experimental data which became available shortly after, in summer 1994 reduced the gap between the exclusive and inclusive determinations of $|V_{cb}|$.

It must be noted, however, that most of the existing theoretical analyses essentially assume that the approximate duality between the actual hadronic amplitudes and the quark-gluon ones sets in already at the excitation energies ~ 0.7 to 1 GeV. In the formfactor analyses it is most obvious in applying the perturbative computations to charm quarks and relying on $1/m_c$ expansion. In computation of the decay widths we, alternatively, invoke the local quark-hadron duality. While there are no experimental indications so far that this is not the case (at least, in the semileptonic physics), the proof is not known either. If that is not true, and duality generally starts only above 1 GeV, most probably one would have to abandon the idea of accurate determination of $|V_{cb}|$ from the exclusive $B \rightarrow D^{(*)}$ transitions. The only option still open will be the inclusive semileptonic decays where the energy release is large, ~ 3.5 GeV. Of course, in such a pessimistic scenario the theoretical precision in $|V_{cb}|$ will hardly be better than 5%.

7 Challenges in the HQE

Concluding the review it is appropriate to mention cases where the heavy quark expansion apparently has problems. It is probably premature to speak of a direct contradiction to experiment; nevertheless, today's question marks carry the seeds of tomorrow's advances. Basically there are two problems where our theoretical understanding is lagging behind. Both are related to nonleptonic decays.

7.1 Semileptonic branching ratio of B and $\Gamma(b \rightarrow c\bar{c}s)$

The theoretical attitude to this problem changes with time. Twenty years ago the parton model gave a prediction $\text{BR}_{\text{sl}}(B) \simeq 13\text{--}15\%$, which was accurate enough according to the existed standards. The variation reflected mostly the choice of quark masses. While the rates of $c \rightarrow c\ell\nu$ and $b \rightarrow c\bar{u}d$ were affected by the choice of m_c in more or less the same way, the rate of the $b \rightarrow c\bar{c}s$ channel decreased much faster when larger masses (closer to the ‘constituent’ rather than short-distance ‘current’ ones) were used. Therefore, $\text{BR}_{\text{sl}}(B)$ increases for larger masses. It should be remembered that the difference $m_b - m_c$ is fixed, so the choices of m_c and m_b are always correlated. Even though the heavy quark symmetry had not been formulated, the latter fact was clearly realized at least in the early 80's [97].

The experimental situation was not very definite and indicated a rather large

$\text{BR}_{\text{sl}}(B)$ which reasonably fitted the ‘larger’ mass option [98]. Since then it became standard to use the larger quark masses, and the parton model prediction for $\text{BR}_{\text{sl}}(B)$ was accepted to be 13–14%, even though a smaller value could be obtained as well. Since the impact of the nonperturbative corrections was completely unknown and presumably significant, $\text{BR}_{\text{sl}}(B)$ was not under intense scrutiny even when the better data became available.

The situation changed when in 1992 Bigi *et al.* showed that there are no $1/m_Q$ nonperturbative corrections to the inclusive widths, both semileptonic and nonleptonic. The leading $1/m_b^2$ nonperturbative effects were readily calculated [18, 62, 54]. They appeared to be suppressed, in particular, as a result of certain cancellations. The overall effect $\Delta_{\text{npt}}\text{BR}_{\text{sl}}(B)$ was found to be about -0.5% ; the nonperturbative effects could not be blamed for a discrepancy any more [99].

This prompted a more careful analysis of the perturbative corrections to the widths. In particular, the $\mathcal{O}(\alpha_s)$ corrections to the nonleptonic $b \rightarrow c\bar{u}d$ width were calculated accounting for the nonzero m_c [100]; this additionally enhanced the nonleptonic width. Later the account for the charm mass was also completed for $b \rightarrow c\bar{c}s$ [101, 102, 103]. Altogether, these $\mathcal{O}(\alpha_s)$ corrections further decreased $\text{BR}_{\text{sl}}(B)$ down to 11–12%. Although this shift naively seems very significant and may raise concerns about the convergence of the perturbative corrections, it actually is not dramatic if one starts with more appropriate short-distance masses, the choice forgotten for historical rather than rational reasons. The relative increase in the nonleptonic width was particularly significant in the $b \rightarrow c\bar{c}s$ channel.

The issue of the semileptonic branching ratio must be considered in conjunction with the charm yield n_c , the number of charm states emerging from B decays. To measure n_c one assigns charm multiplicity *one* to D , D_s , Λ_c and Ξ_c and *two* to charmonia. Zero is assigned to the charmless hadronic final state. It is obvious that

$$n_c \simeq 1 + \text{BR}(B \rightarrow c\bar{c}s\bar{q}) - \text{BR}(B \rightarrow \text{no charm}). \quad (183)$$

Such a joint analysis was motivated already in [18]: the energy release in $b \rightarrow c\bar{c}s$ is rather small, and this can lead to significant duality-violating and higher-order effects. The stability of the perturbative expansion also downgrades. Measuring n_c allows one to effectively exclude this channel from the theoretical calculations.

At present the semileptonic fraction coming both from CLEO and LEP seems to be 1% to 1.5% lower than the ‘preferred’ theoretical expectation. The latter can be decreased playing with the ratio of the quark masses and boosting the effect of perturbative corrections – for example, taking strong coupling at a lower scale. This, however, increases n_c beyond the experimental limits. In principle, allowing n_c to lie in the upper corner of the experimental interval this would almost accommodate the LEP value, but still is somewhat off the CLEO interval for $\text{BR}_{\text{sl}}(B)$ and n_c .

It is quite possible that this discrepancy should be taken seriously, and we would have to admit certain limitations in our ability to compute the nonleptonic B decay width. However, a more thorough analysis is badly needed which would be free from prejudices. The existing compilations rely on the old analysis where, among other things, pole masses of b and c are used as the input. This leads to a number of

problems: the perturbative corrections look too significant, and simply rewriting the first order result in different – but equivalent to order α_s – forms yields noticeably differing numbers. This, somewhat artificial, ambiguity is sometimes used to stretch the limits in a desired direction. Second, using the pole masses for quarks does not allow to immediately constrain their values which are nowadays well known in the proper scheme.

Another place where refinement is needed, is a more accurate determination of the Wilson coefficient for μ_G^2 in the semileptonic fraction. So far all the analyses used the expressions (168) from the original papers [18, 62] where it was obtained in the leading logarithmic approximation only (barring the $b \rightarrow c\bar{c}s$ channel, $\text{BR}_{\text{sl}}(B)$ is modified only via nonfactorizable diagrams appearing to order α_s), being proportional to a small product of two different color coefficients $a_1 a_2$ of the weak nonleptonic Lagrangian. This may be a poor approximation, and the non-log piece coming from the extra gluon with $k \sim m_b$ can dominate the result, as often happens with the LLA expressions in practice. Here there are special reasons to expect significance of such corrections: First, the corresponding diagrams can include gluon-gluon interaction, and such coupling is enhanced by color factors. Second, the LLA term suffers accidental cancellation between the different channels, the fact external to QCD itself. It is improbable that the chromomagnetic interaction itself can lower semileptonic fraction by, say, two percent. However, the existing apparent discrepancy between theory and experiment is lower now than it appeared eight years ago, and this certainly prompts a more careful analysis, to draw the final conclusion.

7.2 Lifetimes of beauty hadrons

As stated before, differences between meson and baryon decay widths arise already in order $1/m_Q^2$. The perturbative corrections to the lifetime ratios are completely absent. The lifetimes of the various mesons get differentiated effectively first in order $1/m_Q^3$. A detailed review can be found in [104, 70, 71].

Because the charm quark mass is not much larger than typical hadronic scale one can expect to make only semi-quantitative predictions on the *charm* lifetimes, in particular for the charm baryons. The agreement of the predictions with the data is good. I would even say it is too good keeping in mind that the $1/m_c$ expansion can hardly be justified. It is difficult to avoid mentioning one particular example. It is generally understood that the large lifetime ratio between charged and neutral D mesons, $\tau_{D^+}/\tau_{D^0} \simeq 2.5$ is due to destructive Pauli Interference in nonleptonic decays of D^+ . However, it is the semileptonic fraction of D^+ which is close to the ‘canonical’ value of 20% obtained by simple-minded counting of the quark decay channels, while $\text{BR}_{\text{sl}}(D^0)$ is much smaller. Yet, if one adds the $1/m_c^2$ shift in the semileptonic fraction due to the chromomagnetic term computed in Ref. [18], the experimental values of the semileptonic fractions are reproduced. Clearly, one should not take such a coincidence too literally since the corrections of order unity discussed here cannot be rigorously justified in the framework of the standard $1/m_Q$ expansion itself.

Table 1: Predictions for Beauty Lifetimes (heavy quark expansion)

Observable	QCD Expectations ($1/m_b$ expansion)	Ref.	Data from [105]
$\tau(B^-)/\tau(B_d)$	$1 + 0.05(f_B/200 \text{ MeV})^2$	[72]	1.070 ± 0.02
$\bar{\tau}(B_s)/\tau(B_d)$	$1 \pm \mathcal{O}(0.01)$	[104]	0.945 ± 0.039
$\tau(\Lambda_b)/\tau(B_d)$	$\gtrsim 0.9$	[104]	0.794 ± 0.053

As far as the *beauty* lifetimes are concerned the $1/m_b$ expansion is to be applicable. Table 1 contains the recent data together with the predictions. The latter were actually made before data (or data of comparable sensitivity) became available.

Data and predictions on the meson lifetimes are nontrivially consistent. Yet even so, a comment is in order for proper orientation. The numerical predictions were based on the assumption of factorization at a typical hadronic scale which is commonly taken as the one where $\alpha_s(\mu_{\text{hadr}}) \simeq 1$. While there is no justification for factorization at $\mu \sim m_b$, there exists ample circumstantial evidence in favor of approximate factorization at a typical hadronic scale. More to the point, the validity of factorization can be probed in semileptonic decays of B mesons in an independent way, as was pointed out in [65].

The possible effect of the nonfactorizable contribution has been discussed in detail in [65, 68], and later in the dedicated paper [71]. They include the nonvalence gluon mechanism discussed long ago in [106] in the simplified language of the quark model. Significant nonfactorizable contributions would in general lead to large effects of Weak Annihilation in D_s mesons where experimentally such effects are quite small. Of course, we cannot reliably treat Γ_D in the $1/m_Q$ expansion, and $\Gamma_{D_s} - \Gamma_{D^+}$ is sensitive to a particular combination of the expectation values of a few four-fermion operators. Nevertheless, this is an indication that the effects due to nonfactorizable expectation values must be suppressed. Later studies based on QCD sum rules and preliminary lattice results reproduced within intrinsic uncertainties the estimates one obtains relying on factorization at the low hadronic scale. The nonfactorizable effects may affect the difference between $\bar{\tau}(B_s)$ and $\tau(B_d)$.

The agreement of the data on B meson lifetimes with experiment is obscured by the apparent conflict for $\tau_{\Lambda_b}/\tau_{B_d}$. To predict the $1/m_Q^3$ corrections to τ_{Λ_b} in the $1/m_b$ expansion one needs to evaluate the baryonic expectation values of two operators,

$$\langle \Lambda_b | \bar{b}b \bar{u}\gamma_0 u | \Lambda_b \rangle, \quad \langle \Lambda_b | \bar{b} \frac{\lambda_a}{2} b \bar{u}\gamma_0 \frac{\lambda_a}{2} u | \Lambda_b \rangle. \quad (184)$$

They do not have the usual factorizable contribution, and their values are rather uncertain. Nevertheless, it was shown that their contributions cannot be too large [107] and the maximal effect in Γ_{Λ_b} does not exceed 10–12%. To achieve larger corrections one would have to go beyond the usual description of baryons where light quarks are “soft”. The reason for such a limitation is rather simple. The four-fermion expectation values, in the language of ordinary quantum mechanics, are squares of the wavefunction at the zero separation between the heavy and the

corresponding light quark:

$$\Psi_{Qq}(0) = \int \frac{d^3\vec{p}}{(2\pi)^3} \Psi_{Qq}(\vec{p}) , \quad (185)$$

but, at the same time, the space integral of $|\Psi_{Qq}|^2$ is limited by unity. A too large value of $\Psi_{Qq}(0)$ means that the wavefunction is sharply peaked at zero separation, which signifies presence of high-momentum modes. The above mentioned general bound agrees with the fact that the constituent quark model estimates typically yield only about 3 to 5% enhancement [108]. A similar conclusion has been reached by the authors of Ref. [109] who analyzed the relevant baryonic matrix elements through QCD sum rules. A dedicated discussion of these questions can be found in Ref. [71].

Recently, the preliminary lattice results for the relevant four-fermion expectation values were reported from the UKQCD collaboration, with the corresponding estimate $\tau_{\Lambda_b}/\tau_{B_d} \simeq 0.90$ [110]. We note here that the $1/m_Q^4$ corrections to the lifetimes can well constitute 30 to 50% of the $1/m_Q^3$ terms. Keeping in mind that the dominant effect in the Λ_b lifetime is Weak Scattering, it is plausible that these effects further enhance the width. (This depends on the corresponding dimension-7 four-fermion operators with derivatives; taking the naive quark model picture one would simply replace m_b^2 in the Wilson coefficient by the square of the average diquark energy in Λ_b , to account for these effects.)

If future more accurate experimental data confirm the present deviation of $\tau_{\Lambda_b}/\tau_{B_d}$ from unity, and refinements of the theoretical estimates do not reveal unexpected enhancement, the most probable explanation of the discrepancy will be violation of local quark-hadron duality in nonleptonic decays. As a matter of fact, its significance in nonleptonic widths is theoretically expected *a priori*. Indeed, the expansion parameter for the widths is not m_b directly but rather the energy release which is noticeably smaller in $b \rightarrow c$. Moreover, the preasymptotic corrections depend on the concrete form of the weak interaction involved. For the four-fermion interaction they are enhanced: the arguments based on the high power of mass m_Q^5 in the decay rate [22] suggest that the actual scale parameter is smaller, $\propto E_{\text{rel}}/5$. In the semileptonic decays this does not deteriorate the expansion since it is automatically protected by the heavy quark symmetry when m_c increases.

For the nonleptonic decays the heavy quark symmetry does not generally apply, and at insufficient energy release one expects significant violations of duality. The real problem here is that the few leading terms in the expansion have been evaluated and did not indicate the expansion blowing up at the level of 20%.

8 Heavy Quark Expansion and Violations of Local Duality

A large number of applications of the heavy quark theory are based on quark-hadron duality, more exactly, its local implementation. Although this notion becomes ex-

act at asymptotically high energies, at finite energy scale certain deviations must be present. How fast duality sets in and how large are these deviations are important questions. These and similar questions are among most difficult. Yet they are important for phenomenology: determinations of $|V_{cb}|$ we discussed rely on the assumption of local duality.

The notion of duality in general terms was first introduced in the early days of QCD in Ref. [111] but not pursued for quite some time. It was simply regarded as a problem of how the observables, say $\sigma(e^+e^- \rightarrow \text{hadrons})$ as a function of energy computed through quarks (and gluons) can be equated to the actual cross section. The former predict small smooth deviations from the would be free quarks production, while the latter has well-shaped resonant structure reflecting quarks permanently confined in colorless hadrons. Since such hadronization effects are nonperturbative in nature, the question of local duality is tightly related to treatment of strong interactions beyond perturbation theory. This was not really available in the early days of QCD; confinement was simply assumed not to affect averaged cross sections “significantly”.

Nowadays we have in our disposal methods which are applicable – at least in principle – to quantify nonperturbative effects in a number of observables, including those of the actual Minkowski world. These are quantities amenable to study via the OPE. Local quark-hadron duality was given a new consideration a few years ago by M. Shifman [112], who related its violations to the asymptotic nature of the power expansion in inverse large energy scale provided by the ‘practical’ OPE. These ideas were later reiterated and developed in a number of papers, in particular in connection to heavy quark physics (see, e.g., Refs. [67, 74]).

The basic observation can be illustrated on the example of heavy quark decay widths. The “practical” OPE yields the width in the power expansion

$$\frac{\Gamma_{H_Q}}{\Gamma_Q} = A_0 + \frac{A_1}{m_Q} + \frac{A_2}{m_Q^2} + \dots \quad (186)$$

If the series in $1/m_Q$ were convergent (to the actual ratio), Γ_{H_Q} would have been an analytic function of m_Q above a certain mass m_0 pointing to the onset of the *exact* local parton-hadron duality. The actual Γ_{H_Q} is definitely non-analytic at any threshold (whether or not the amplitude vanishes at the threshold). Thus, the ‘radius of convergence’ cannot correspond to the mass smaller than the threshold mass. Since in the actual QCD the thresholds exist at arbitrary high energy, the power expansion in Eq. (186) can be only asymptotic, with formally zero radius of convergence in $1/m_Q$.

In practice, the true threshold singularities are expected to be strongly suppressed at large energies, and the corresponding uncertainties in the OPE series quite small. Eventually they are expected to be exponentially suppressed, though, possibly, starting at larger energies. In the intermediate domain they can decrease as a certain power and must *oscillate*. This reflects the peculiarity of the Minkowski world. The terms left aside by the ‘practical’ OPE are exponentially suppressed in the Euclidean domain, $\sim \exp -\sqrt{Q^2}$ where Q^2 generically denotes the square of the

large momentum scale. Continuing this expression to Minkowski domain and taking discontinuity to determine transition probabilities, as discussed in Sects. 5 and 6, we observe oscillating rather than exponentially suppressed effects.

As was illustrated in Ref. [74], the power expansions like Eq. (186) are meaningful even beyond the power suppression where the duality-violating oscillations show up. In the case of the heavy quark widths where mass m_Q cannot be varied in experiment, the size of the duality-violating component may set the practical bound for calculating the widths. Thus, it is important to have an idea about its size. At the same time one should always include the leading QCD effects to the partonic expressions, rather than compare the actual observable with the bare quark result. In the model considered in Ref. [74], incorporating the power corrections from the practical OPE suppressed the apparent deviations by more than an order of magnitude.

It is useful to keep in mind that violations of local duality, although so far poorly understood dynamically, are not arbitrary and must obey constraints following from the OPE. They cannot be blamed, for instance, for the systematic excess or systematic depletion of decay probabilities; the actual width can only oscillate around the OPE predictions as a function of energy or quark mass, or the difference has to fade out exponentially.

Although the simplest illustration of the asymptotic nature of the decay width $1/m_Q$ expansion and related violations of local duality given above follows from the presence of hadronic thresholds, violation of local duality is a more universal phenomenon that is *not* directly related to existence of hadronic resonances nor even confinement itself. This has been illustrated in Ref. [67] by the example of soft instanton effects that do not lead, at least at small instanton density, to quark confinement – but do indeed generate computable oscillating duality-violating contributions to the total decay rates.

One can parallel the OPE with expanding the interaction between quarks and gluons (and their related propagation) at small distances set up by the inverse energy scale in the problem. This is transparent in toy quantum mechanical problems where the terms in the OPE series can be traced back to the Taylor expansion of the potential $V(x)$ at $x \rightarrow 0$. The behavior of $V(x)$ in the vicinity of $x=0$ does not exhaust the problem, however. The spectrum of final states crucially depends on the asymptotics of the potential at large x ; this is analogous to the resonances affecting local duality. There are more intriguing mechanisms associated with the finite-distance singularities of QCD interactions – the example is provided by instantons. The interested reader can find the inspiring discussion in the dedicated paper by Shifman in this volume [113].

While conceptual grounds for violation of local duality have become more clear, its dynamical origin in actual QCD is still not well understood. What type of physics lies behind the finite- x singularities if they are relevant? How to quantify effects of resonances at high energies? We do not have answers yet. The instanton-based model [67], for example, while capturing correctly the gross features, clearly falls short in describing the size of the effects, at least under the standard assumptions. In all considered cases, actually, the effects of local duality violation turn out very

small in the asymptotic domain of large energies or quark masses, much below phenomenologically significant level. For practical purposes, we would like to know the limitations imposed by duality in the domain of intermediate energies where it *a priori* can be sizeable. The use of analytic methods can be limited here.

Trying to get insights into the possible magnitude of violations of local duality in semileptonic decays of heavy quarks, recent paper [50] studied their numerical significance in the exactly solvable two-dimensional 't Hooft model. The model has built-in “hard” confinement related to linear Coulomb potential in 1+1 dimensions. Its spectrum consists of towers of infinitely narrow resonances. As was mentioned above, duality violation is not necessarily related to existence of resonances. Nevertheless, the intuition remains that resonance dominance is not “favorable” for the OPE, and problems might show up, for instance, through a delayed numerical onset of duality, in that the approximate equality of the OPE predictions and the actual decay widths may set in only after a significant number of thresholds has been passed. To address such issues, the 't Hooft model seems to represent the most certain testing ground for local duality in the domain of decays of moderately heavy quarks.

Contrary to naive expectations, surprisingly accurate duality was found between the (truncated) OPE series for Γ_{sl} and the actual decay widths. The deviations were suppressed to a very high degree almost immediately after the threshold for the first excited final state hadron is passed. No suspected delay in the onset of duality was found. Remarkably, the ‘practical’ OPE turns out to be the most efficient way to obtain numerical predictions for the total semileptonic width in the model to a high accuracy hardly accessible to direct computations, down to rather low quark masses.

Some of the duality-violating features observed in those studies have natural explanations. At fixed energy release $m_b - m_c$ the magnitude of the deviations is smaller if m_b, m_c are both large than if they are both small. This is expected, since in the former case the heavy quark symmetry for the elastic amplitude additionally enforces approximate duality even when no expansion in large energy release can be applied. However, at fixed m_b the duality violation decreased rapidly as m_c decreases, in full accord with the OPE where the higher order terms are generally suppressed by powers of $1/(m_b - m_c)$. This is clearly a *dynamical* feature that goes beyond heavy quark symmetry *per se*, the quality of which deteriorates as m_c decreases.

To the extent the numerical findings of Ref. [50] can be transferred to real QCD, violation of local duality in the total semileptonic widths of B mesons should not be an issue. The scale of duality violation lied far below the phenomenologically accessible limits, and could not affect the credibility of extracting $|V_{cb}|$ or $|V_{ub}|$.

In reality there are, of course, essential differences between the two theories, including those aspects that are expected to be important for local duality (for a discussion, see Ref. [66]). Although many seem to optimistically suggest that duality violation is more pronounced in the 't Hooft model than for actual heavy flavor hadrons, some differences may still work in the opposite direction. In $D=2$ there are no dynamical gluons, nor a chromomagnetic field that in $D=4$ provides a

significant scale of nonperturbative effects in heavy flavor hadrons. Likewise, there is no spin in $D = 2$, and no corresponding P -wave excitations of the spin- $\frac{3}{2}$ light degrees of freedom which play an important role in $D = 4$.

Two-dimensional QCD neither has long perturbative “tails” of actual strong interactions suppressed weakly (by only powers of logs of the energy scale). In $D = 2$ the perturbative corrections are generally power-suppressed, as follows from the dimension of the gauge coupling. As discussed in Ref. [66], it is conceivable that the characteristic mass scale for freezing out the transverse gluonic degrees of freedom is higher than in the “valence” quark channels. This would imply a possibly higher scale for onset of duality in α_s/π corrections to various observables.

Regardless of these differences, it was demonstrated that presence of resonance structure *per se* is not an obstacle for fine local quark-hadron duality tested in the context of the OPE. In the ’t Hooft model resonances themselves do not demand a larger duality interval. As soon as the mass scale of the states saturating the sum rules in a particular channel (quark or hybrid) has been passed, the decay width can be well approximated numerically by the expansion stemming from the OPE.

The same pattern is expected in QCD. To get a better idea regarding the onset of local duality, at least in semileptonic decays, it is important to study the saturation of the heavy quark sum rules, in particular the static SV sum rules discussed in Sect. 4. Are the values of μ_π^2 and μ_G^2 we use reached at the energy scale of 0.7 to 1 GeV and right above we can use the perturbative description to account for higher states, – or we need to go higher in energy for that? Experimental study of actual $b \rightarrow c$ transitions can give the answer. There are ideas how this can be studied on the lattices. The question of local duality in nonleptonic decays of beauty particles at present remains largely unknown.

9 Conclusions and Outlook

Heavy quark theory is probably one of the youngest branches of the QCD tree, yet it has become mature. Many practically important problems that laid dormant for years are now tractable. The development of the heavy quark theory, often initiated in the quest for the most accurate extraction of electroweak properties of quarks, enriched the library of available methods in QCD itself. At the same time, many natural limitations of QCD show up in the heavy quark expansion as well. Quark confinement is not completely understood yet, the infrared part of dynamics is more parameterized than solved. Therefore, every new result relying on the general properties of the QCD interactions brings in a theoretical value. The success of the last decade was mainly related to incorporating Wilson’s approach to the heavy quark expansion. It allowed to quantify the deviations from the heavy quark symmetry in many important cases, and led to a number of accurate predictions not appealing to the symmetry itself. The theory of preasymptotic corrections to the decay widths is one of such assets, whether or not b quark is heavy enough for precision applications of the heavy quark expansion.

A clear-cut recent manifestation of the power of the heavy quark theory is the framework it provides for determination of $|V_{cb}|$. It is remarkable that the values of $|V_{cb}|$ that emerged from exploiting two theoretically complementary approaches relying on quite different assumptions, are very close. The progress was not for free: it became possible only due to essential refinements of the available theoretical tools. They prompted us, in particular, that the zero recoil $B \rightarrow D^*$ formfactor F_{D^*} is probably closer to 0.9, with much larger deviations from the symmetry limit than previously expected. The predicted decrease in F_{D^*} brought more accurate experimental data which appeared later in a good agreement with $|V_{cb}|$ determined from $\Gamma_{\text{sl}}(B)$. The confidence in the latter method, in turn required many improvements beyond theory of nonperturbative effects as well, including accurate evaluation of the perturbative corrections. The latter story, although eventually successful, was far from simple (for the historical perspective see, e.g. Ref. [2] and references therein). Sharpening the numerical predictions was not possible without clarifying the nature of the heavy quark mass entering theoretical expression, the questions which long caused confusion in the literature. It is encouraging that now there is a consensus regarding also the numerical value of the running b quark mass – it is extracted by different groups from the Υ sum rules using the state of the art NNLO resummation of the perturbative effects. At the same time, a consistent theoretical definition of such a low-scale mass required investigating the dipole radiation in the non-Abelian gauge theory, a very general – although still somewhat theoretical – physical phenomenon.

There are still problems to solve, ahead. Some of them are notoriously difficult, like violations of local duality. Others may look at once only technical, for example, developing the constructive Wilsonian OPE including perturbation theory, where the soft parts have to be removed from all diagrams preserving such basic properties of QCD interactions as gauge invariance. Like in the previous years, we may expect new links in the chain of theoretical advances to emerge from both types of studies.

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